Quantum Computing HW1

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Question 1 - Inner Products, Angles and Measurement of a single qubit

Let $|\psi\rangle = \cos(\pi/8)|0\rangle + \sin(\pi/8)|1\rangle$.

a)

We are to write $|\psi\rangle$ using the standard notation. Let us recall:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \; ; \; |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Thus:

$$|\psi\rangle = \cos(\pi/8)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin(\pi/8)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(\pi/8) \\ \sin(\pi/8) \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2+\sqrt{2}}}{2} \\ \frac{\sqrt{2-\sqrt{2}}}{2} \end{pmatrix}$$

b)

We are to verify that $|||\psi\rangle||_2 = \sqrt{\langle\psi|\psi\rangle} = 1$:

$$|||\psi\rangle||_2 = \sqrt{\langle\psi|\psi\rangle} = \sqrt{(|\psi\rangle)^\dagger|\psi\rangle} = \sqrt{\begin{pmatrix} \cos(\pi/8) & \sin(\pi/8) \end{pmatrix}\begin{pmatrix} \cos(\pi/8) \\ \sin(\pi/8) \end{pmatrix}} =$$
$$\sqrt{\cos^2(\pi/8) + \sin^2(\pi/8)} = \sqrt{1} = 1$$

c)

The inner product between $|\psi\rangle$ and $|0\rangle$:

$$\langle\psi|0\rangle = (|\psi\rangle)^\dagger|0\rangle = \begin{pmatrix} \cos(\pi/8) & \sin(\pi/8) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \cos(\pi/8) = \frac{\sqrt{2+\sqrt{2}}}{2}$$

The angle between $|\psi\rangle$ and $|0\rangle$:

$$\theta = \cos^{-1}\left(\frac{(|\psi\rangle)^\dagger|0\rangle}{|||\psi\rangle|| \cdot |||0\rangle||}\right) = \cos^{-1}(\cos(\pi/8)) = \pi/8$$

since by the definition of an angle between two vectors - $\theta \in [0, \pi]$.

d)

We are given a quantum state which is either $|0\rangle$ or $|\psi\rangle$ uniformly at random. We win if we are able to identify correctly the state that we are given, and we loose otherwise. We are to calculate the winning probabilities of the given strategies.
Let us observe that the given quantum state is chosen uniformly at random between $|\psi\rangle$ and $|0\rangle$, and thus the probability for choosing each is $1/2$. So we have:

$$\mathbb{P}[\text{the quantum state is } |0\rangle] = \mathbb{P}[\text{the quantum state is } |\psi\rangle] = 1/2$$

(a)

Measure the qubit in the computational basis. Answer '0' if the outcome is '0', and answer 'ψ' if the outcome is 1.
The winning probability of the strategy is:

$$\mathbb{P}[\text{win}] = 1 - \mathbb{P}[\text{lose}]$$

where $\mathbb{P}[\text{win}]$ is the winning probability and $\mathbb{P}[\text{lose}]$ is the losing probability. Let us calculate the probability $\mathbb{P}[\text{lose}]$. Let us observe that losing can only happen if the outcome is '0', yet the quantum state is $|\psi\rangle$ and not $|0\rangle$. Thus:

$$\mathbb{P}[\text{lose}] = \mathbb{P}[\text{the outcome is '0'} \wedge \text{the quantum state is } |\psi\rangle]$$

and by relation for conditional probability:

$$\begin{aligned} & \mathbb{P}[\text{the outcome is '0'} \wedge \text{the quantum state is } |\psi\rangle] \\ = \; & \mathbb{P}[\text{the outcome is '0'} \mid \text{the quantum state is } |\psi\rangle] \cdot \mathbb{P}[\text{the quantum state is } |\psi\rangle] \end{aligned}$$

where we calculated the second term. By definition, the first term is the square of the absolute value of the coefficient of $|0\rangle$ in the expression for $|\psi\rangle$, meaning:

$$\mathbb{P}[\text{the outcome is '0'} \mid \text{the quantum state is } |\psi\rangle] = |\cos(\pi/8)|^2 = \cos^2(\pi/8)$$

since $\cos^2(\pi/8) > 0$. Plugging in all the values, we have:

$$\mathbb{P}[\text{win}] = 1 - \frac{\cos^2(\pi/8)}{2} = \frac{6-\sqrt{2}}{8} \approx 0.573$$

(b)

Apply the Hadamard gate and then measure the qubit in the computational basis. Answer 'ψ' if the outcome is '0', and answer '0' if the outcome is '1'.
Let us apply the Hadamard gate to both $|\psi\rangle$ and $|0\rangle$:

$$\begin{aligned} H|\psi\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \cos(\pi/8) \\ \sin(\pi/8) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos(\pi/8) + \sin(\pi/8) \\ \cos(\pi/8) - \sin(\pi/8) \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\Big(\cos(\pi/8) + \sin(\pi/8)\Big)|0\rangle + \frac{1}{\sqrt{2}}\Big(\cos(\pi/8) - \sin(\pi/8)\Big)|1\rangle \\ H|0\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = |+\rangle \end{aligned}$$

Now, for this strategy, we have:

$$\begin{aligned} \mathbb{P}[\text{win}] = \mathbb{P}\Big[&\Big(\text{the outcome is '0'} \wedge \text{the quantum state is } |\psi\rangle\Big) \vee \\ &\Big(\text{the outcome is '1'} \wedge \text{the quantum state is } |0\rangle\Big)\Big] \end{aligned}$$

Since these two events are disjoint, we have:

$$\mathbb{P}\Big[\Big(\text{the outcome is '0'} \wedge \text{the quantum state is } |\psi\rangle\Big) \vee \Big(\text{the outcome is '1'} \wedge \text{the quantum state is } |0\rangle\Big)\Big]$$
$$= \mathbb{P}[\text{the outcome is '0'} \wedge \text{the quantum state is } |\psi\rangle] + \mathbb{P}[\text{the outcome is '1'} \wedge \text{the quantum state is } |0\rangle]$$

Now, let us calculate the last two values. The first of them:

$$\begin{aligned} & \mathbb{P}[\text{the outcome is '0'} \wedge \text{the quantum state is } |\psi\rangle] \\ = \; & \mathbb{P}[\text{the outcome is '0'} \mid \text{the quantum state is } |\psi\rangle] \cdot \mathbb{P}[\text{the quantum state is } |\psi\rangle] \end{aligned}$$

Using the same reasoning from the first strategy, in this case we calculate the conditional probability using the result from the Hadamard gate:

$$\mathbb{P}[\text{the outcome is '0'} \mid \text{the quantum state is } |\psi\rangle] = \left(\frac{1}{\sqrt{2}}\Big(\cos(\pi/8) + \sin(\pi/8)\Big)\right)^2$$
$$= \frac{1}{2}\Big(\cos^2(\pi/8) + 2\cos(\pi/8)\sin(\pi/8) + \sin^2(\pi/8)\Big) = \frac{1}{2}\Big(1 + \sin(\pi/4))\Big) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)$$

Now for the second term:

$$\begin{aligned} & \mathbb{P}[\text{the outcome is '1'} \wedge \text{the quantum state is } |0\rangle] \\ = \; & \mathbb{P}[\text{the outcome is '1'} \mid \text{the quantum state is } |0\rangle] \cdot \mathbb{P}[\text{the quantum state is } |0\rangle] \end{aligned}$$

where:

$$\mathbb{P}[\text{the outcome is '1'} \mid \text{the quantum state is } |0\rangle] = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$$

Plugging in all the values we get:

$$\mathbb{P}[\text{win}] = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right) \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}\left(2 + \frac{1}{\sqrt{2}}\right) \approx 0.676$$

One can see that the second strategy has a greater probability of success.

Question 2 - Tensor Product State

Let $|\alpha\rangle = a|0\rangle + b|1\rangle, |\beta\rangle = c|0\rangle + d|1\rangle$, where $a, b, c, d \in \mathbb{C}$.

a)

Let us define the basis $B = \{|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle\}$. We are to write $|\alpha\rangle \otimes |\beta\rangle$ in the basis B using both the Dirac notation and the usual matrix notation. By the Dirac notation:

$$\begin{aligned} |\alpha\rangle \otimes |\beta\rangle \quad &= \big(a|0\rangle + b|1\rangle\big) \otimes \big(c|0\rangle + d|1\rangle\big) \\ &= ac|0\rangle \otimes |0\rangle + ad|0\rangle \otimes |1\rangle + bc|1\rangle \otimes |0\rangle + bd|1\rangle \otimes |1\rangle \end{aligned}$$

Using this result, let us now consider the the usual matrix notation. For two qubits, let us recall:

$$|0\rangle \otimes |0\rangle = |00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \; ; \; |0\rangle \otimes |1\rangle = |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \; ; \; |1\rangle \otimes |0\rangle = |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \; ; \; |1\rangle \otimes |1\rangle = |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Using this, we have:

$$\begin{aligned} |\alpha\rangle \otimes |\beta\rangle \quad &= ac|0\rangle \otimes |0\rangle + ad|0\rangle \otimes |1\rangle + bc|1\rangle \otimes |0\rangle + bd|1\rangle \otimes |1\rangle \\ &= ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle \\ &= ac \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + ad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + bc \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + bd \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix} \end{aligned}$$

b)

We are to calculate the standard inner product between $|\alpha\rangle$ and $|\beta\rangle$:

$$\begin{aligned} \langle\alpha|\beta\rangle \quad &= (|\alpha\rangle)^\dagger |\beta\rangle = \big(a|0\rangle + b|1\rangle\big)^\dagger \big(c|0\rangle + d|1\rangle\big) \\ &= \left(a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)^\dagger \left(c \begin{pmatrix} 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} a \\ b \end{pmatrix}^\dagger \begin{pmatrix} c \\ d \end{pmatrix} \\ &= \begin{pmatrix} a^* & b^* \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = a^* c + b^* d \end{aligned}$$

where a^* and b^* are the complex conjugates of a and b respectively.

c)

We are to calculate the norm of $|\alpha\rangle$:

$$|||\alpha\rangle||_2 = \sqrt{\langle\alpha|\alpha\rangle} = \sqrt{(|\alpha\rangle)^\dagger |\alpha\rangle} = \sqrt{\begin{pmatrix} a^* & b^* \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}} =$$
$$\sqrt{a^* a + b^* b} = \sqrt{|a|^2 + |b|^2}$$

where a^* and b^* are the absolute values of a and b respectively.
Now we are to prove that $|||\alpha\rangle \otimes |\beta\rangle||_2 = |||\alpha\rangle||_2 \cdot |||\beta\rangle||_2$. Starting from the LHS:

$$
\begin{aligned}
|||\alpha\rangle \otimes |\beta\rangle||_2 = || \begin{pmatrix} ac & ad & bc & bd \end{pmatrix}^T ||_2 &= \sqrt{\begin{pmatrix} (ac)^* & (ad)^* & (bc)^* & (bd)^* \end{pmatrix} \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}} \\
&= \sqrt{|ac|^2 + |ad|^2 + |bc|^2 + |bd|^2} = \sqrt{|a|^2|c|^2 + |a|^2|d|^2 + |b|^2|c|^2 + |b|^2|d|^2} \\
&= \sqrt{|a|^2\big(|c|^2 + |d|^2\big) + |b|^2\big(|c|^2 + |d|^2\big)} = \sqrt{\big(|a|^2 + |b|^2\big)\big(|c|^2 + |d|^2\big)} \\
&= \sqrt{|a|^2 + |b|^2}\sqrt{|c|^2 + |d|^2} = |||\alpha\rangle||_2 \cdot |||\beta\rangle||_2
\end{aligned}
$$

d)

Let $|\psi\rangle = \frac{1}{\sqrt{2}}\big(|01\rangle + |10\rangle\big)$. We are to prove or disprove the claim that $|\psi\rangle$ is a tensor product state. Let us disprove the claim by assuming it holds and reaching a contradiction. Thus, let us assume that $|\psi\rangle$ is a tensor product state. Thus, there exist two vectors $|r\rangle = r_0|0\rangle + r_1|1\rangle$ and $|s\rangle = s_0|0\rangle + s_1|1\rangle$ such that $|\psi\rangle$ can be written in the following manner:

$$
\begin{aligned}
|\psi\rangle \quad &= |r\rangle \otimes |s\rangle = (r_0|0\rangle + r_1|1\rangle) \otimes (s_0|0\rangle + s_1|1\rangle) \\
&= r_0 s_0|00\rangle + r_0 s_1|01\rangle + r_1 s_0|10\rangle + r_1 s_1|11\rangle
\end{aligned}
$$

Thus, by the definition of $|\psi\rangle$ we must have that:

$$
\begin{aligned}
& r_0 s_1 = \frac{1}{\sqrt{2}} \land r_1 s_0 = \frac{1}{\sqrt{2}} \land r_0 s_0 = 0 \land r_1 s_1 = 0 \\
& \rightarrow r_0 \neq 0 \land s_1 \neq 0 \land r_1 \neq 0 \land s_0 \neq 0 \land \big(r_0 = 0 \lor s_0 = 0\big) \land \big(r_1 = 0 \lor s_1 = 0\big)
\end{aligned}
$$

thus we have a contradiction.

e)

Let $|\psi'\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle - |01\rangle\big)$. We are to prove or disprove the claim that $|\psi'\rangle$ is a tensor product state. Let us prove the claim by showing that there exist two vectors $|k\rangle$ and $|t\rangle$ such that their tensor product is equal to $|\psi'\rangle$. Let $|k\rangle = k_0|0\rangle + k_1|1\rangle$ and $|t\rangle = t_0|0\rangle + t_1|1\rangle$ such that:

$$
\begin{aligned}
|\psi'\rangle \quad &= |k\rangle \otimes |t\rangle = (k_0|0\rangle + k_1|1\rangle) \otimes (t_0|0\rangle + t_1|1\rangle) \\
&= k_0 t_0|00\rangle + k_0 t_1|01\rangle + k_1 t_0|10\rangle + k_1 t_1|11\rangle \\
&= \frac{1}{\sqrt{2}}\big(|00\rangle - |01\rangle\big)
\end{aligned}
$$

We result in the following system of equations:

$$
\begin{aligned}
k_0 t_0 &= \frac{1}{\sqrt{2}} \\
k_0 t_1 &= -\frac{1}{\sqrt{2}} \\
k_1 t_0 &= 0 \\
k_1 t_1 &= 0
\end{aligned}
$$

from the first two equations we get $k_0, t_0, t_1 \neq 0$ and from the last two equations we get $(k_1 = 0 \lor t_0 = 0) \land (k_1 = 0 \lor t_1 = 0)$. Thus, since $t_0 \neq 0 \land t_1 \neq 0$ we'll get that $k_1 = 0$. Setting $k_0 = 1, t_0 = \frac{1}{\sqrt{2}}, t_1 = -\frac{1}{\sqrt{2}}$ - one has a solution to this set of equations and thus there exist such vectors $|k\rangle$ and $|t\rangle$ - meaning the state $|\psi'\rangle$ is a tensor product state.

Question 3 - Operations and Measurements of Multiple Qubits

Let $|\psi\rangle = \frac{1}{\sqrt{6}}\big(i|00\rangle - 2|10\rangle + |11\rangle\big)$.

a)

We are to write $|\psi\rangle$ using the standard notation. Using the standard notation for two qubits displayed in the last question:

$$|\psi\rangle = \frac{i}{\sqrt{6}}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + 0\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \frac{-2}{\sqrt{6}}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + \frac{1}{\sqrt{6}}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix}i\\0\\-2\\1\end{pmatrix} := \begin{pmatrix}a_{00}\\a_{01}\\a_{10}\\a_{11}\end{pmatrix}$$

b)

We measure both the qubits of the state $|\psi\rangle$ in the basis B defined in the last question. We are to calculate the various probabilities of the result. By the same reasoning mentioned in the previous questions:

$$\mathbb{P}[\text{the result is } 00] = |a_{00}|^2 = \left|\frac{i}{\sqrt{6}}\right|^2 = \frac{1}{6}$$
$$\mathbb{P}[\text{the result is } 01] = |a_{01}|^2 = 0$$
$$\mathbb{P}[\text{the result is } 10] = |a_{10}|^2 = \left|\frac{-2}{\sqrt{6}}\right|^2 = \frac{2}{3}$$
$$\mathbb{P}[\text{the result is } 11] = |a_{11}|^2 = \left|\frac{1}{\sqrt{6}}\right|^2 = \frac{1}{6}$$

c)

We measure only the right qubit of the state $|\psi\rangle$. We are to calculate the probabilities of the outcomes 0 and 1. As seen in class, the probability of measuring 0 in the right cubit is the sum of the probabilities of measuring 00 and 10, and probability of measuring 1 in the right cubit is the sum of the probabilities of measuring 01 and 11. Thus:

$$\mathbb{P}[\text{the right cubit is } 0] = |a_{00}|^2 + |a_{10}|^2 = \frac{1}{6} + \frac{2}{3} = \frac{5}{6}$$
$$\mathbb{P}[\text{the right cubit is } 1] = |a_{01}|^2 + |a_{11}|^2 = 0 + \frac{1}{6} = \frac{1}{6}$$

As we saw in class, when measuring the right cubit - the left cubit collapses to a quantum state with regard to what is the outcome of the measurement of the right cubit. Assuming we measured '0' in the right cubit - the left cubit collapses to (in Dirac and standard notations):

$$\frac{1}{\sqrt{|a_{00}|^2 + |a_{10}|^2}}\begin{pmatrix}a_{00}\\a_{10}\end{pmatrix} = \frac{1}{\sqrt{\frac{1}{6} + \frac{2}{3}}}\begin{pmatrix}\frac{i}{\sqrt{6}}\\\frac{-2}{\sqrt{6}}\end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix}i\\-2\end{pmatrix} = \frac{1}{\sqrt{5}}\big(i|0\rangle - 2|1\rangle\big)$$

d)

We are now asked what happens to the state if we apply a Hadamard gate on the left qubit (before doing any measurement). Let us then apply a Hadamard gate on the left cubit:

$$H \otimes I_{2\times 2}|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \frac{1}{\sqrt{6}} \begin{pmatrix} i \\ 0 \\ -2 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{12}} \begin{pmatrix} i-2 \\ 1 \\ i+2 \\ -1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{12}} \Big((i-2)|00\rangle + |01\rangle + (i+2)|10\rangle - |11\rangle \Big)$$