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Project 18-40

SNIC Bifurcation and its Applications to MEMS

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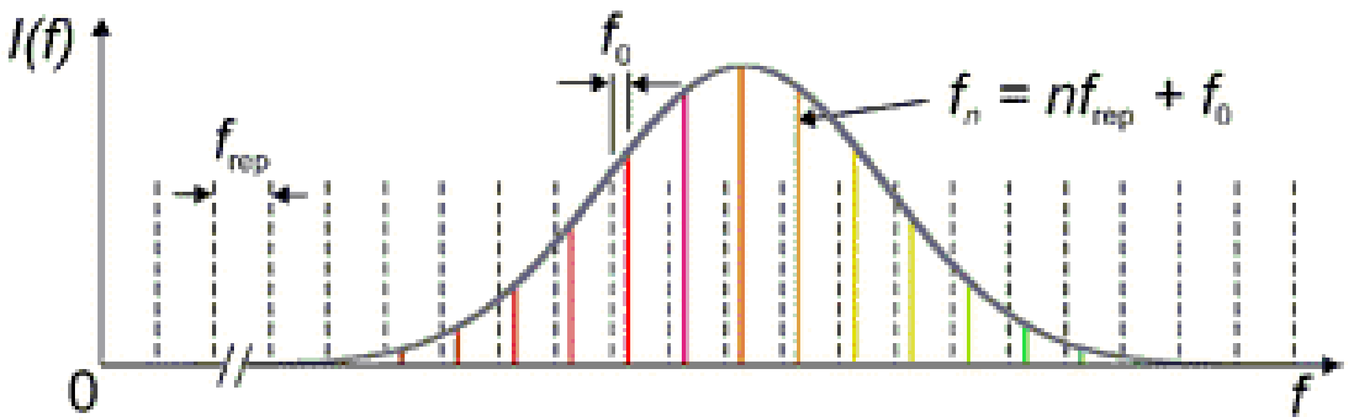
Introduction

Generation of a Frequency Comb

A set of discrete equally spaced frequencies

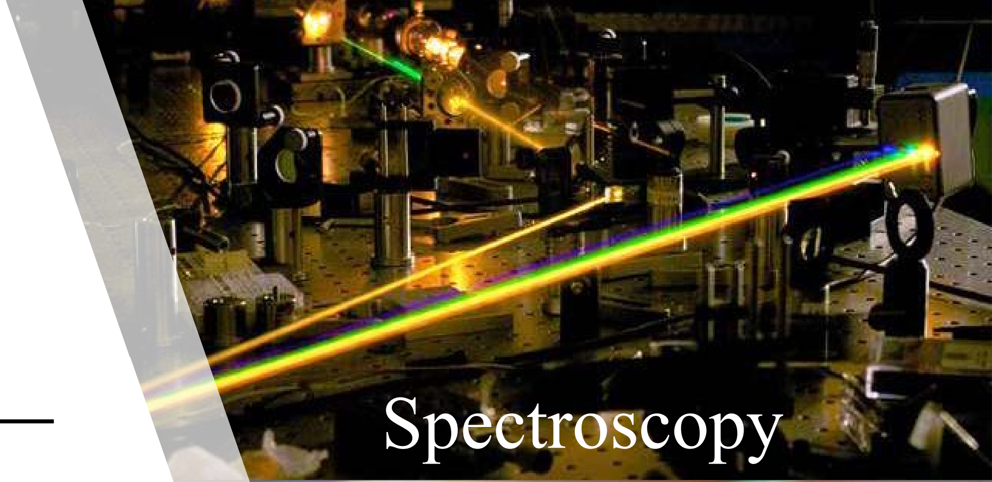


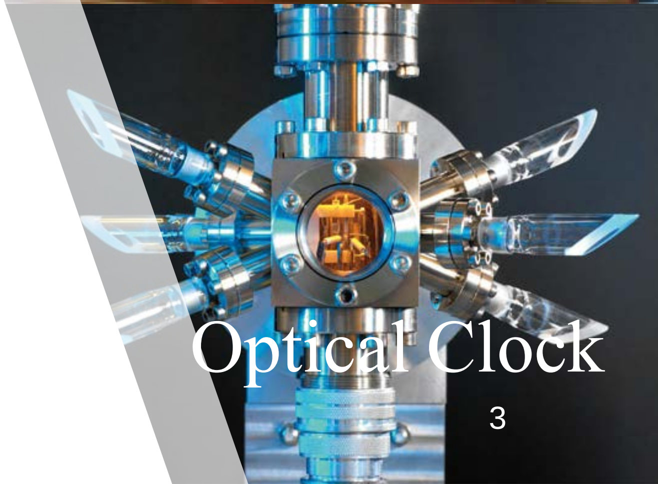
$$f_n \equiv f_0 + nf_r$$

Applications

What can we do with it?

- Transition from one frequency range to another
- Optical and Atomic clocks
- Precision Spectroscopy
- Precision microwave generation





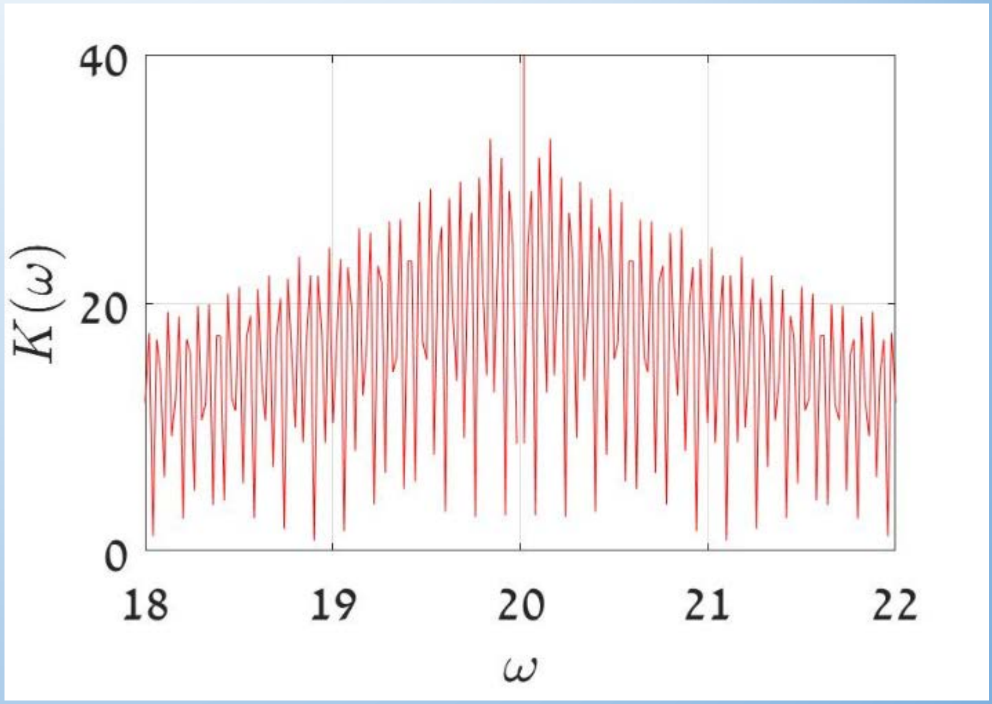
Motivation

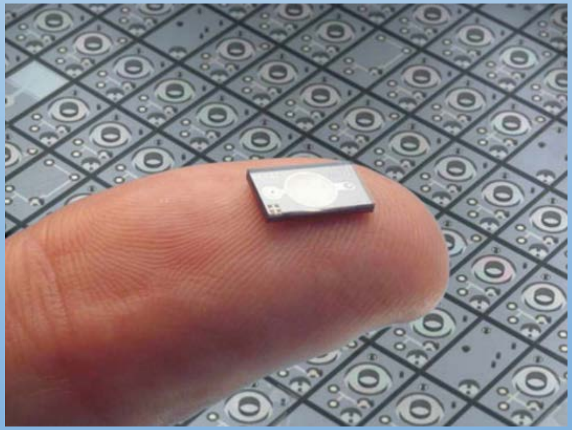
Why mechanically?

- Electrical systems produce noise – corresponding to a poor comb
- Optical systems are expensive to implement
- A mechanical model is more precise

Project Goals

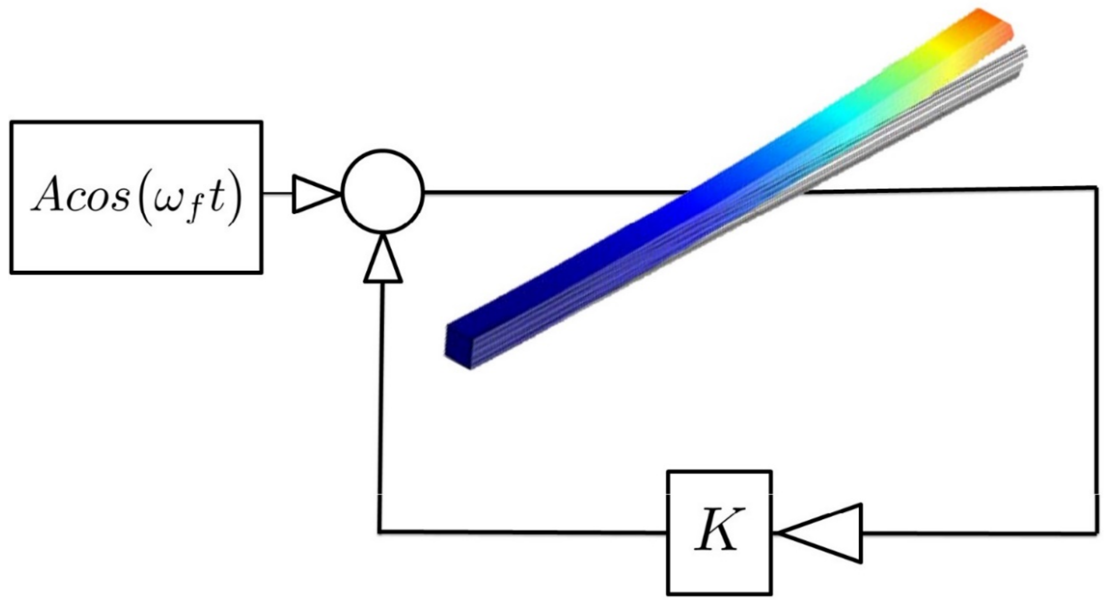
- Producing a frequency comb by mechanical means
- Using a simple dynamic model
- Verifying the model's validity
- Providing insight on usage with MEMS





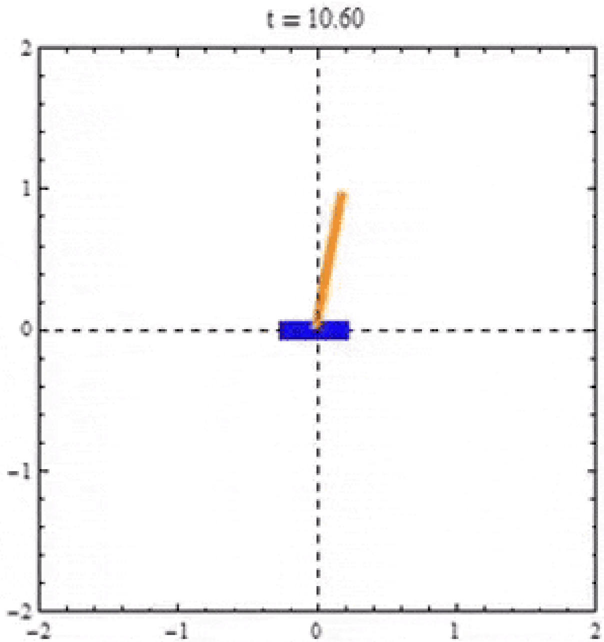
- Vibration analysis of a MEMS beam exhibiting non-linear dynamics for a formation of a limit cycle
- Use of closed loop control to maintain the beam at steady amplitude
- External perturbation of the beam
- Causing a SNIC bifurcation
- Forming the desired frequency comb

Methods of Execution



Closed-loop oscillator

What is a Bifurcation?

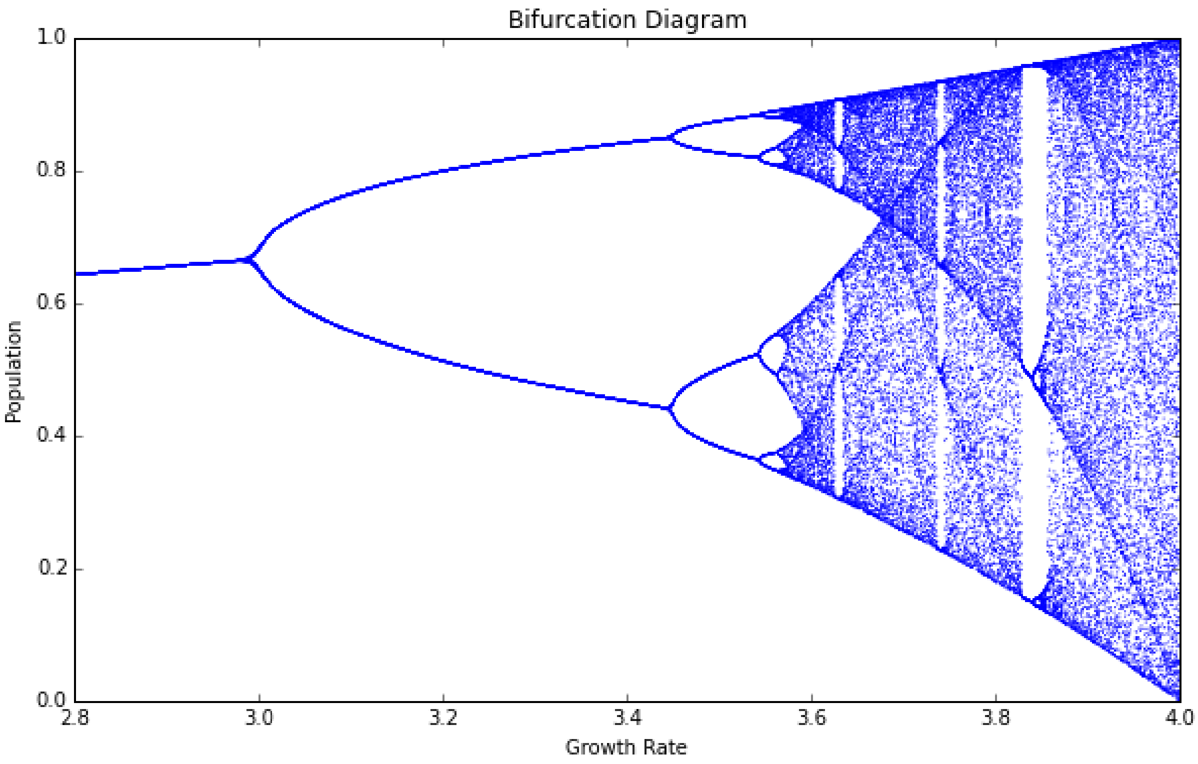


Inverted pendulum exhibiting a bifurcation

An abrupt qualitive change in a behavior of a dynamical system due to a change in one of its governing parameters

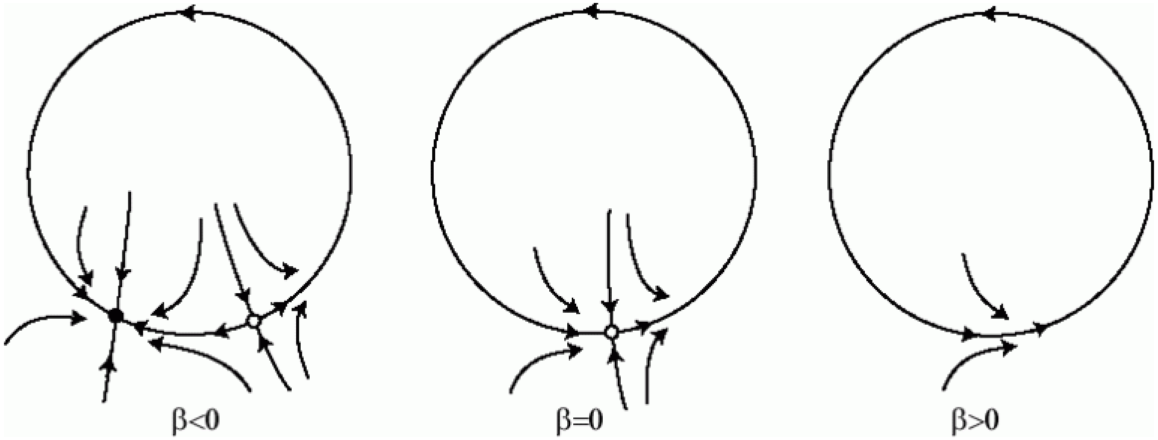
Bifurcation Example
The Logistic Map

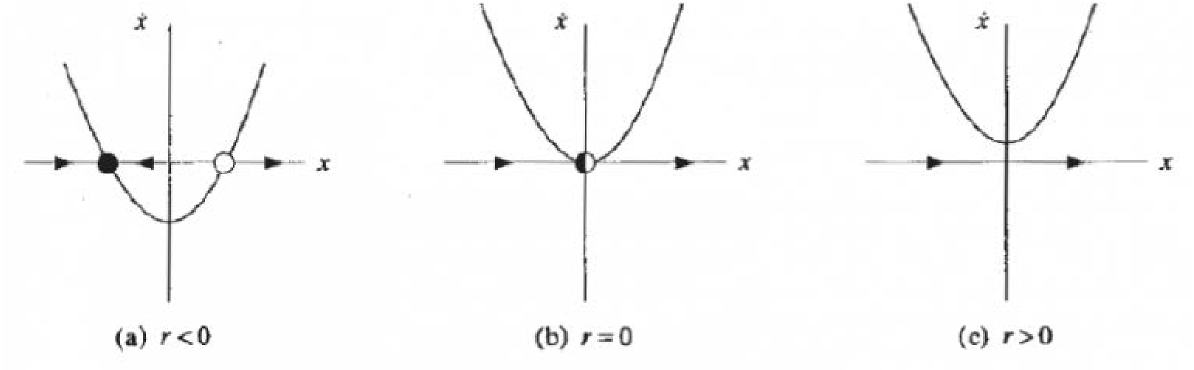
$$x_{n+1} = r x_n (1 - x_n)$$



Models population growth and decay using an iterative equation governed by r – the growth rate

SNIC Bifurcation

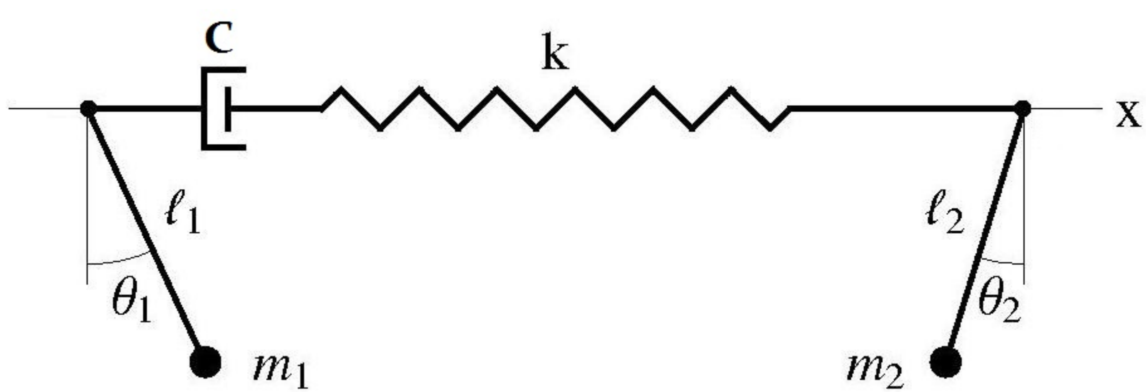




Saddle-Node on Invariant Circle is a bifurcation where two equilibrium points collide and cause the system to settle on an infinite limit cycle

How will SNIC help us get the comb?

- Driver frequency slightly detuned from the beam's eigenfrequency
- Due to effects of injection pulling, a SNIC bifurcation will occur for certain physical parameters of the beam
- In the bifurcation critical point, where the oscillators are close to locking – the frequency comb will appear



Mathematical Modeling

A model from Ron Lifshitz and M. C. Cross. “Nonlinear Dynamics of Nanomechanical and Micromechanical Resonators”

$$\rho A \frac{\partial^2 u(x,t)}{\partial t^2} + 2\zeta \frac{\partial u(x,t)}{\partial t} - \tau[u(x,t)] \frac{\partial^2 u(x,t)}{\partial x^2} + EI \frac{\partial^4 u(x,t)}{\partial x^4}$$
$$= \left[A \cos(\omega_d t) + \psi[u(x,t)]\right] \delta(x - x_0)$$

The beam is taken to be clamped-clamped so the boundary conditions are

$$u(0,t) = u(l,t) = \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(l,t) = 0$$

Spatial Function Solution

We assume:

- Variable Separation
- Single Mode Assumption for the first mode

Yielding the shape function of the beam's first mode

$$X_1(s) = 0.618 \cdot \Big[\sin(s) - \sinh(s)\Big] - 0.629 \cdot \Big[\cos(s) - \cosh(s)\Big]$$

$$s \in [0, 4.75]$$

Temporal Function Solution

Normalizing with respect to time and length results in the following equation for the time dynamics

$$\frac{d^2T}{d\tau^2} + 2\tilde{\zeta}\frac{dT}{d\tau} + T + T^3 = \Phi\cos\left(\frac{\omega_d}{\omega_n}\tau\right) + \Gamma\frac{\frac{dT}{d\tau}}{\left|\frac{dT}{d\tau}\right|}$$

Second derivative with respect to normalized time

Damping term

Linear term

Duffing non-linear term

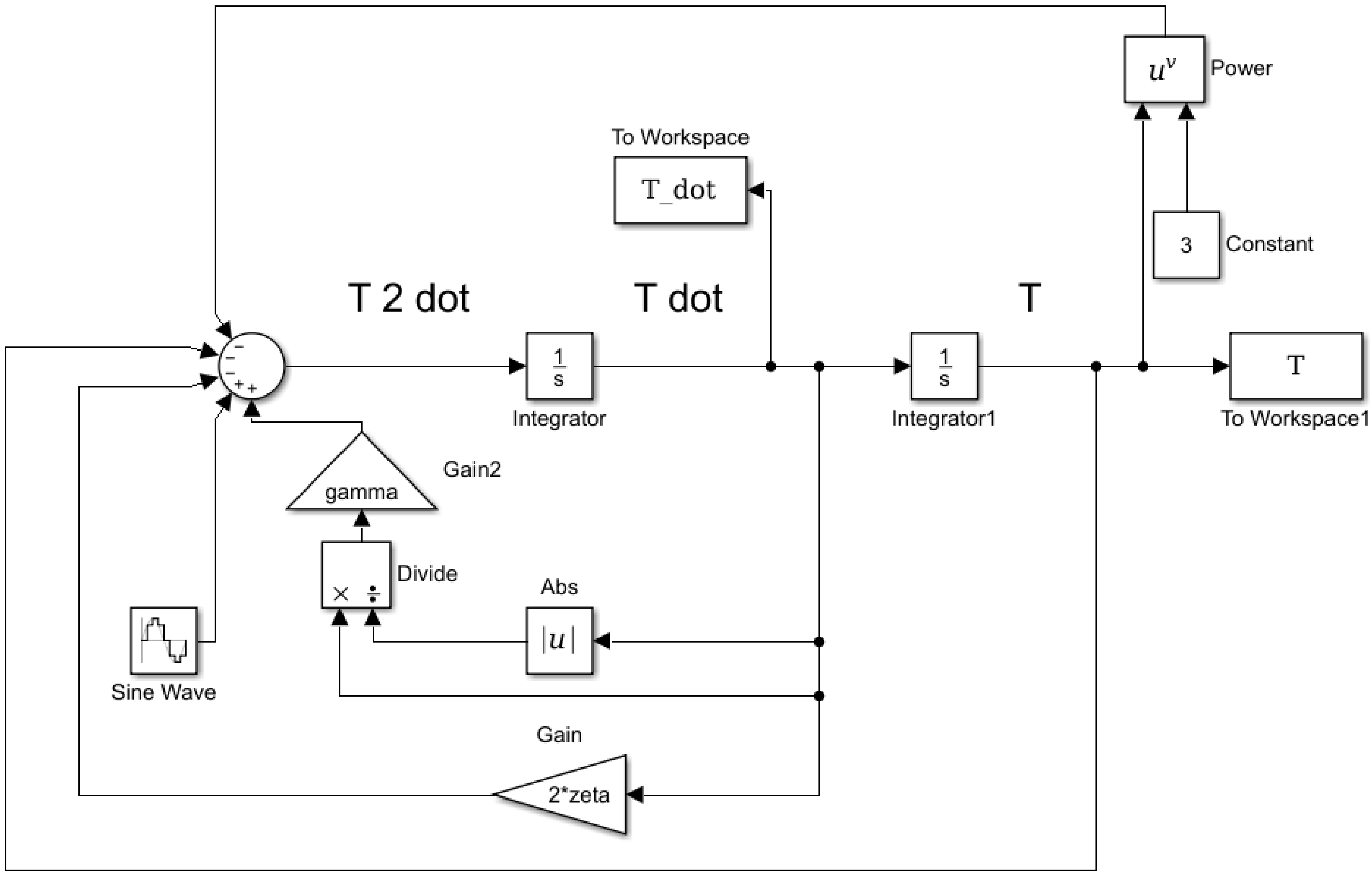
External driver

Closed loop control function

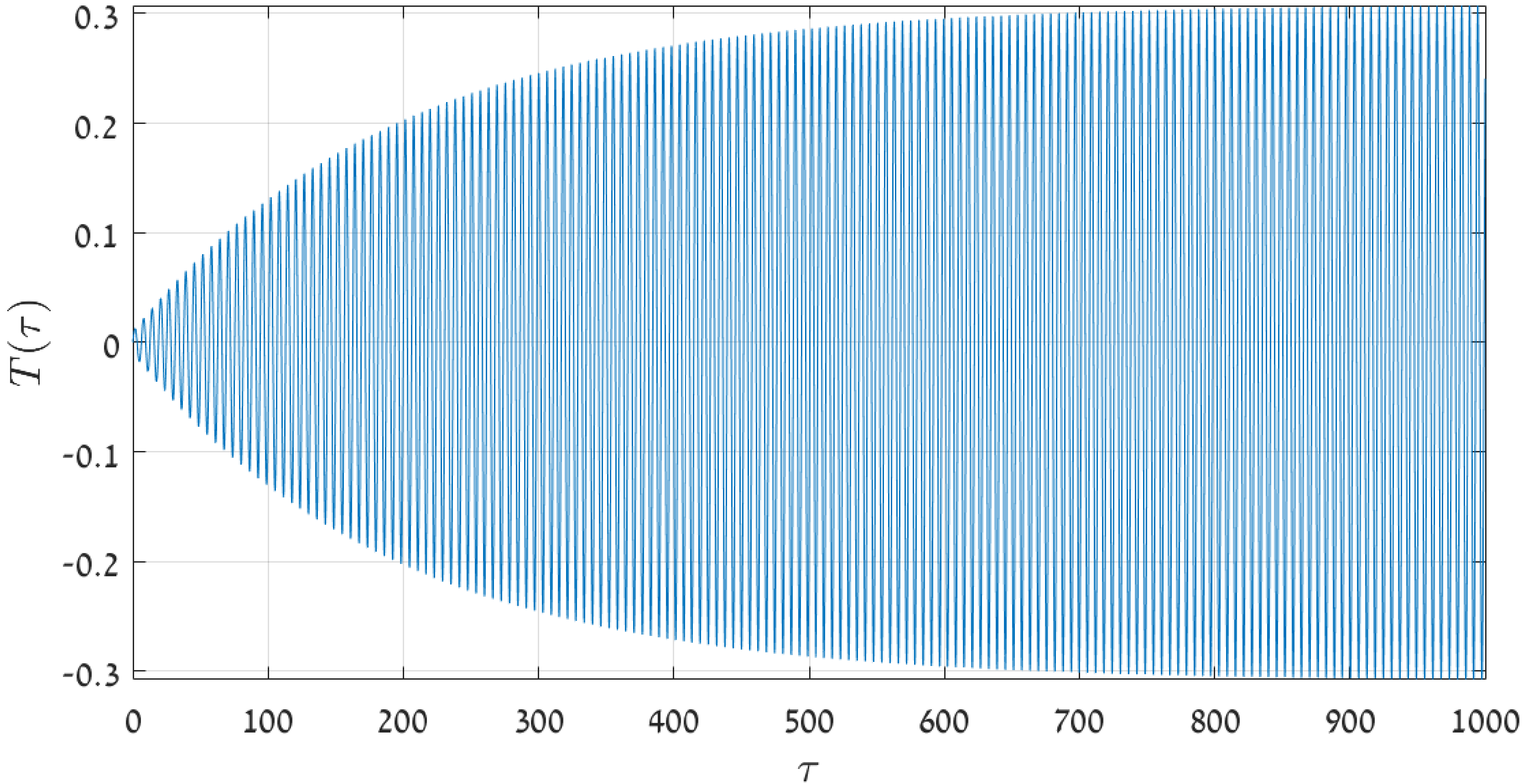
Using a model from Seshia, we take the Signum function to cancel out the damping

$$\psi = \Gamma\frac{\frac{dT}{d\tau}}{\left|\frac{dT}{d\tau}\right|} \longrightarrow \boxed{a_{ss} = \frac{2\Gamma}{\pi\tilde{\zeta}}}$$

Validation of the Model

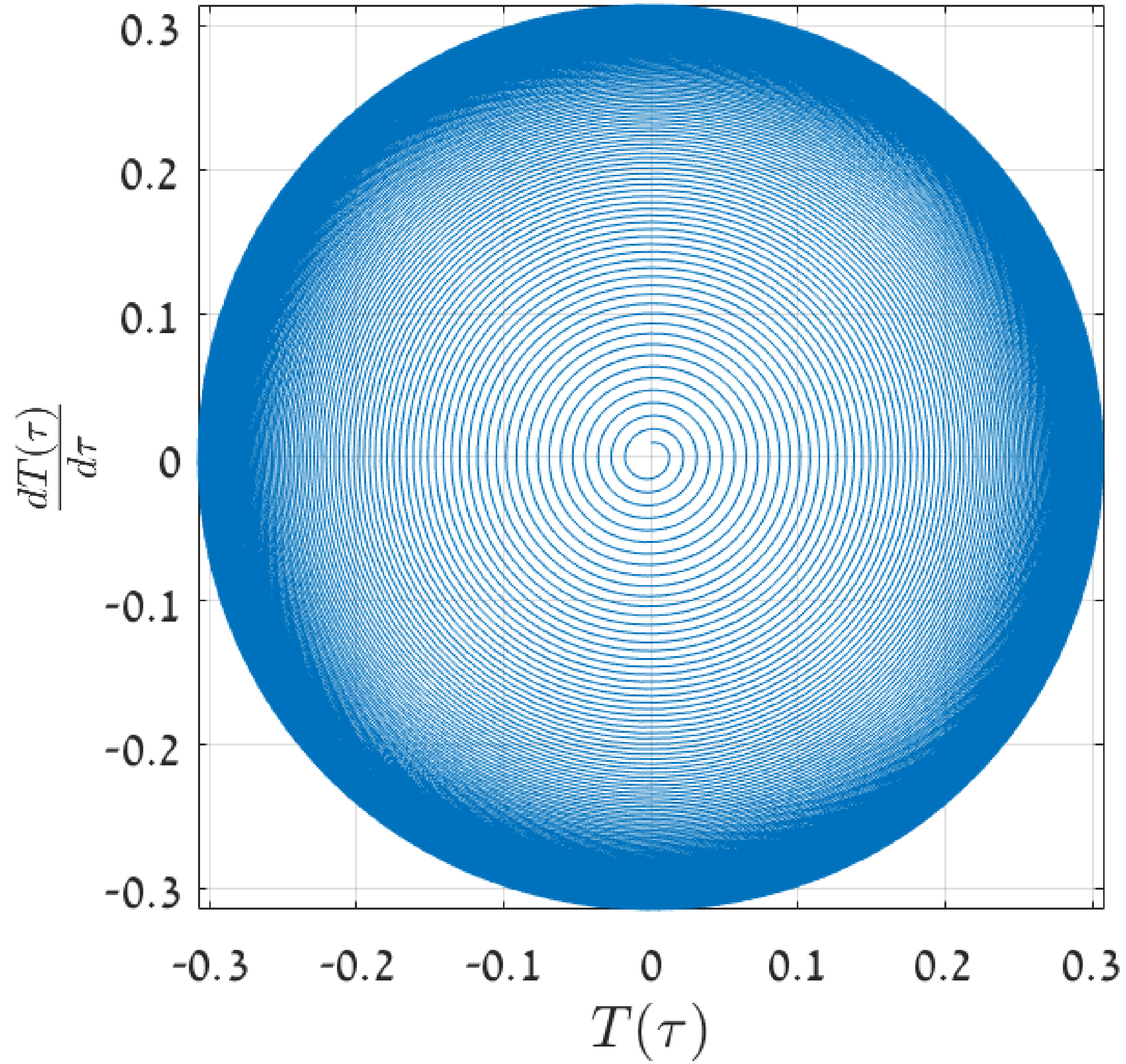


The final equation Simulink model



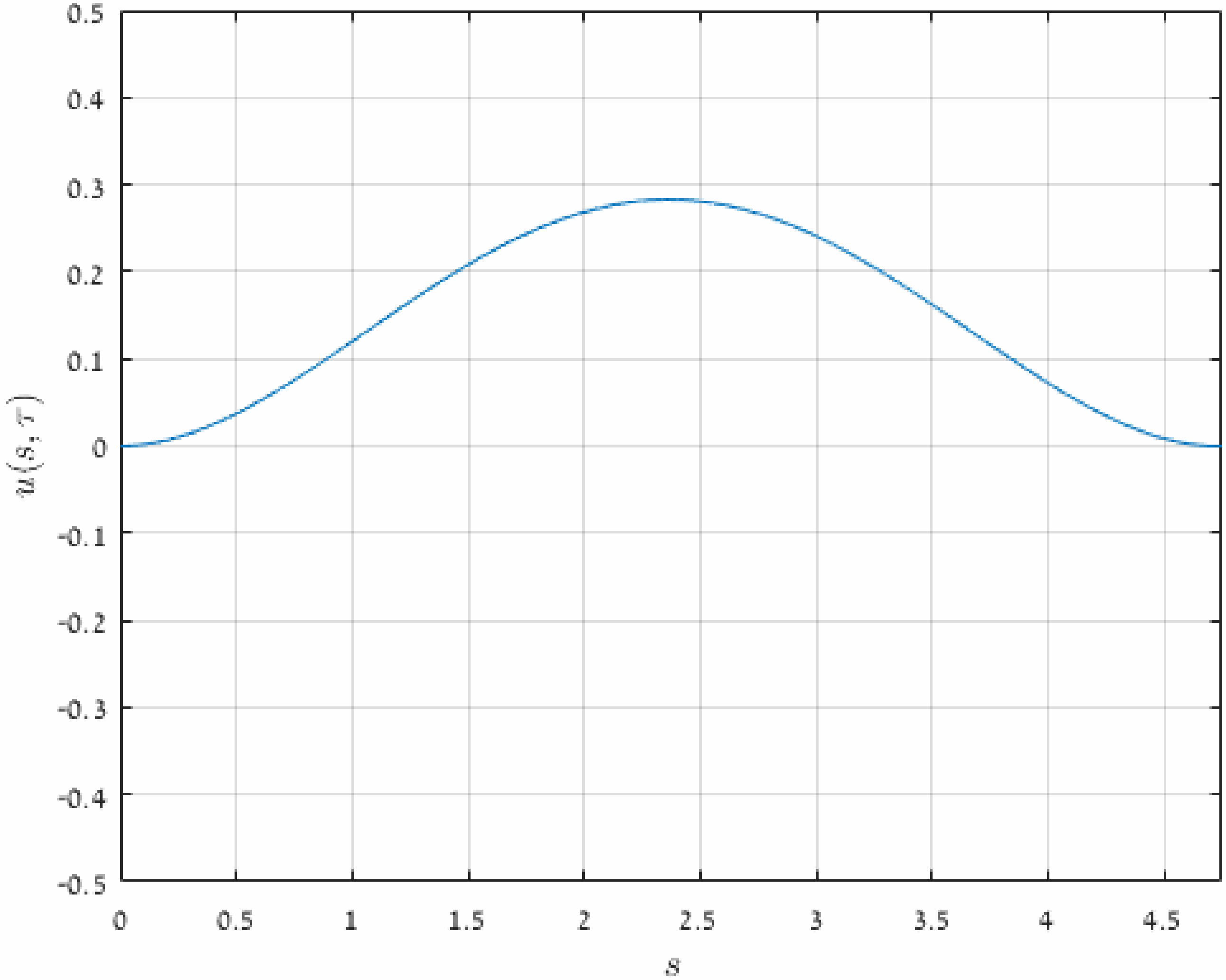
SS Amplitude Simulations 1

Taking $\Gamma = \frac{\tilde{\zeta}}{2}$ we get a period of $\frac{1}{\pi} \approx 0.3$



SS Amplitude Simulations 2

Phase space shows a limit cycle of $\approx$ 0.3



Solution for $u(x, t)$

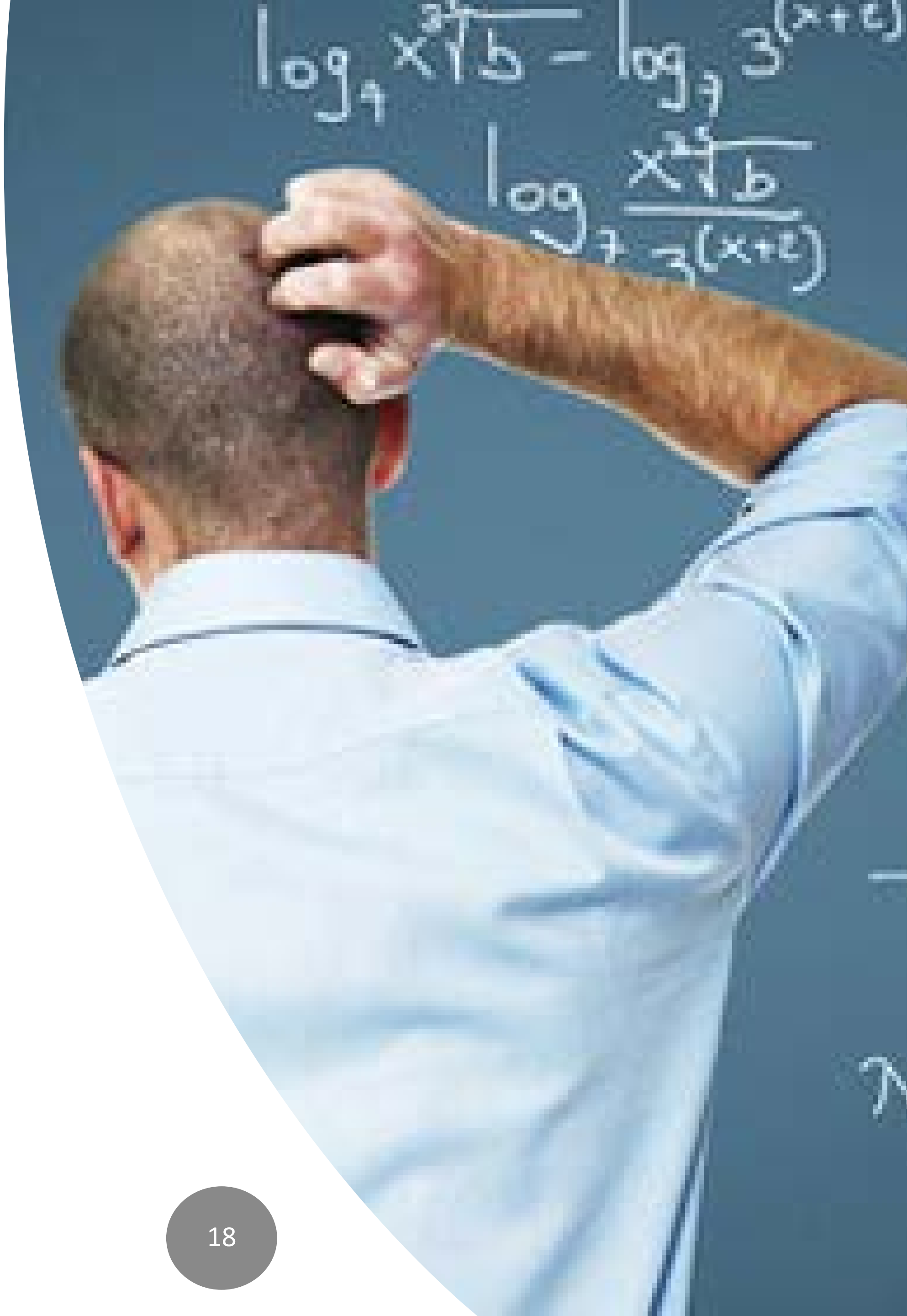
Using the solution for the spatial and temporal functions

Mid Results

So we get a steady limit cycle, and with a small driver we can get the SNIC!

Now we just have to solve the equation to get the dynamics of the beam and derive the phase pull

But.. Even the simple Duffing equation is not solvable



Slow Evolution Model

Using a model of Ashwin A. Seshia for the temporal function

$$T(\tau) = a(\tau)\cos\left[\frac{\omega_d}{\omega_n}\tau + \theta(\tau)\right]$$

Where the functions obey

$$\dot{a} = \frac{2\Gamma}{\pi} - \tilde{\zeta}a - \Phi\sin(\theta)$$

$$\dot{\theta} = \Delta\omega + \frac{3\omega_n}{8\omega_d}a^2 - \frac{\Phi}{a}\cos(\theta)$$

The Adler Equation

Assuming the amplitude converges rapidly, the dynamics is given by the phase function ϕ, which can be reduced to

$$\dot{\phi} \approx B[K - \sin(\phi)]$$

Where $\mathrm{K} = \frac{\Delta\omega}{B}$, $\Delta\omega$ is the phase offset and B is a physical parameter of the beam

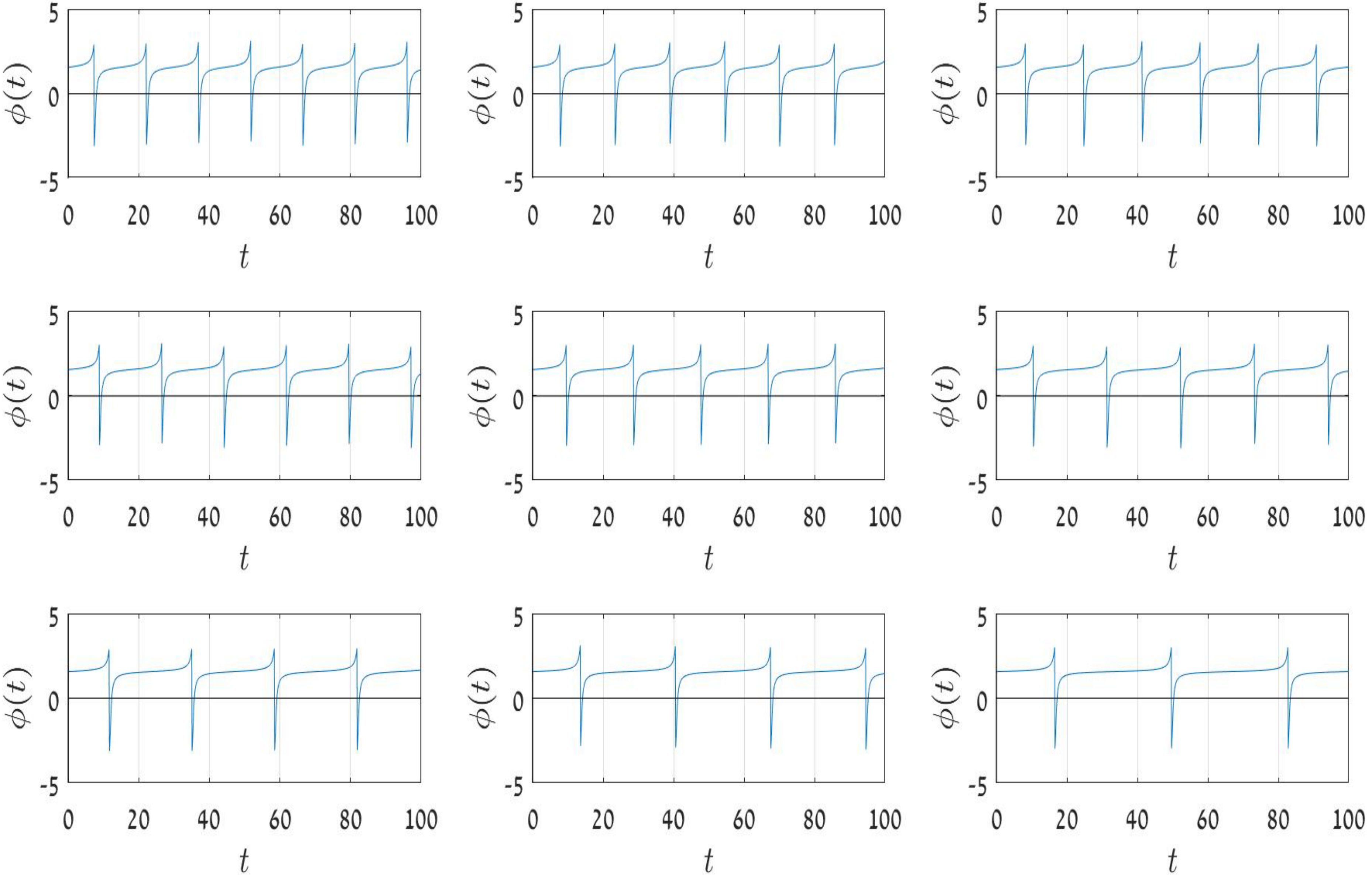
This Adler equation has a closed analytical solution!

Simulating Adler's equation with Matlab for values that obey

$$\lim_{K \to 1^+} \phi(t)$$

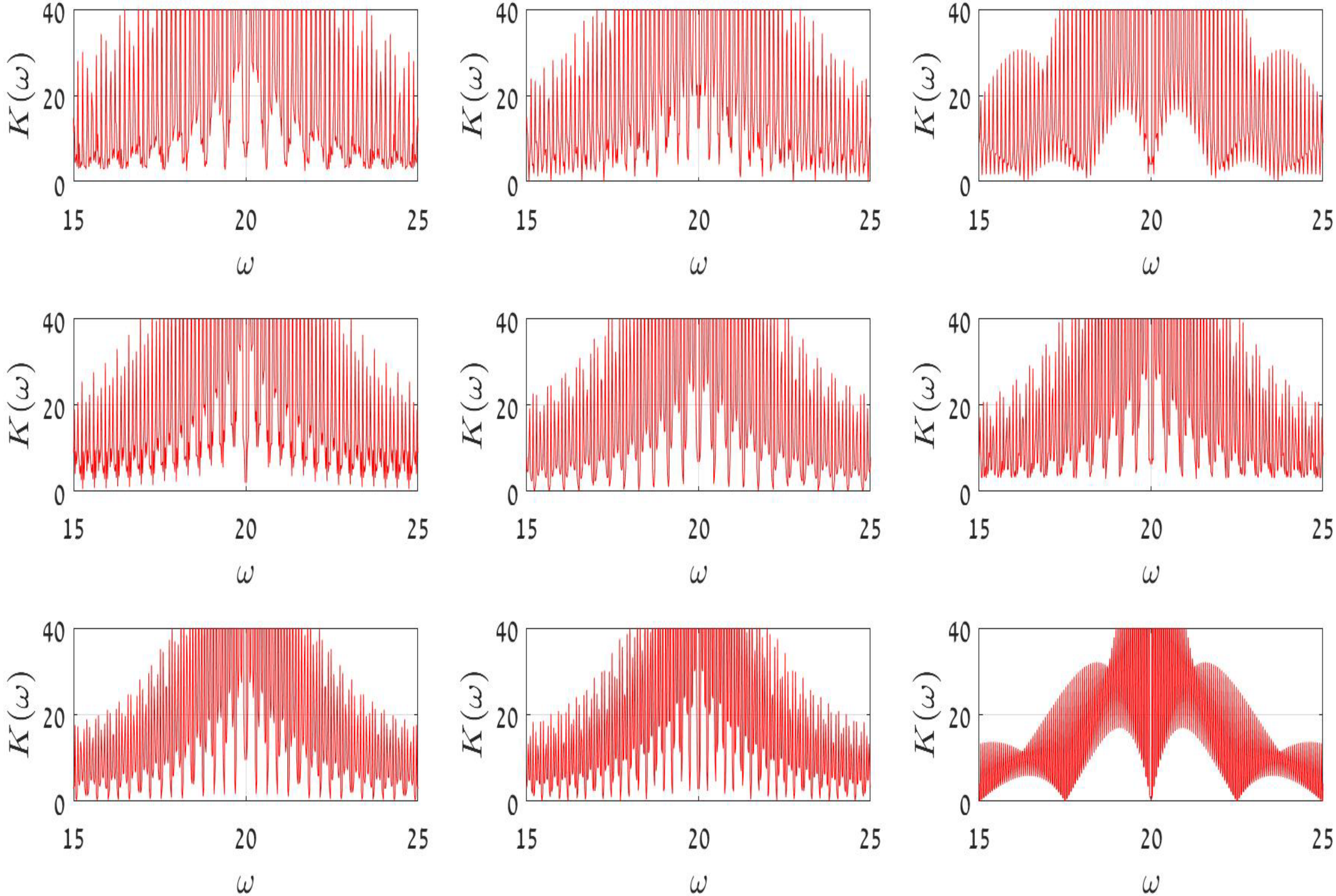
will yield the desired comb!

Phase Time Response



$B = 3\,; K\epsilon[1.002, 1.01]$

Phase Frequency Response



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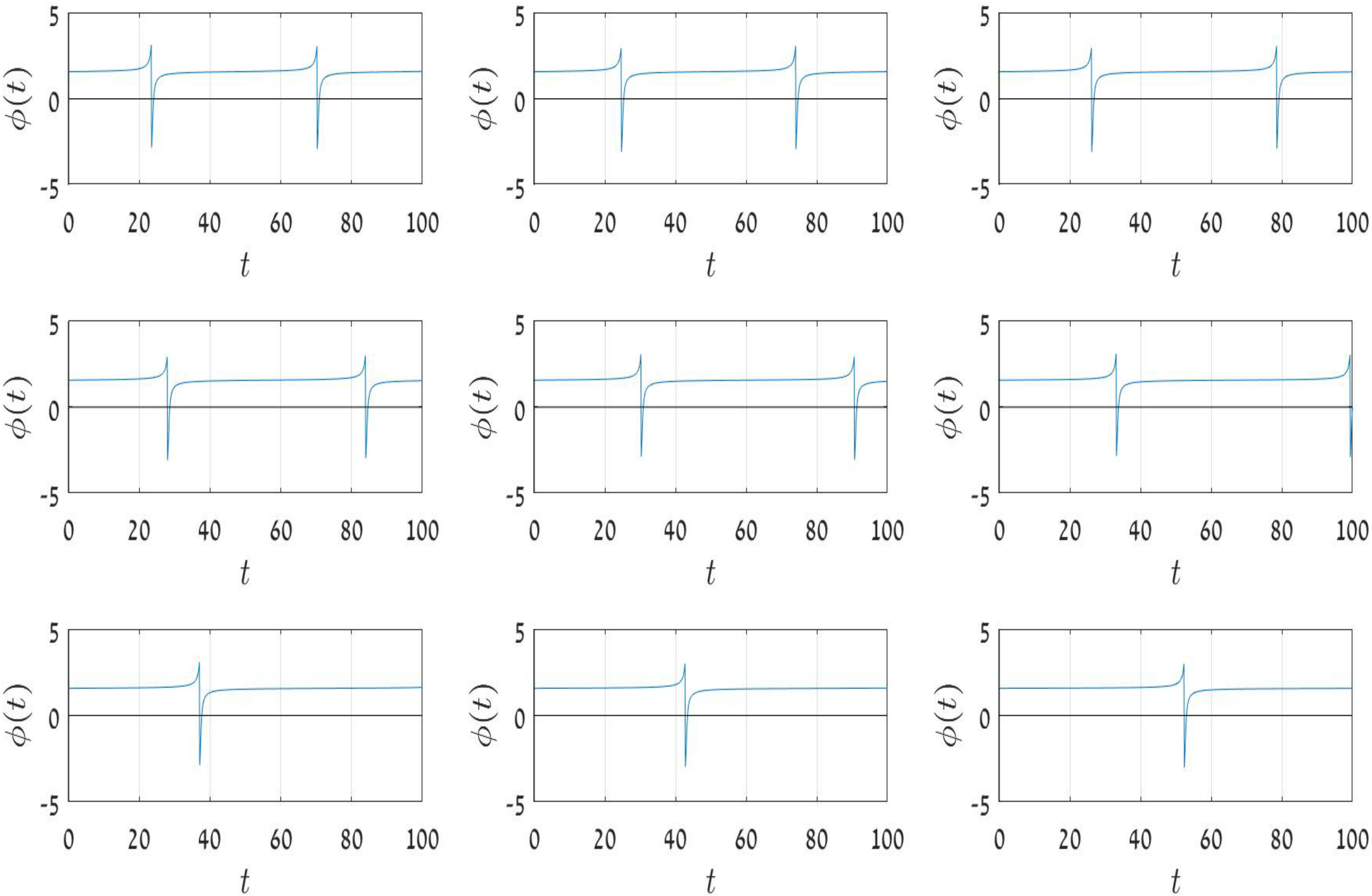
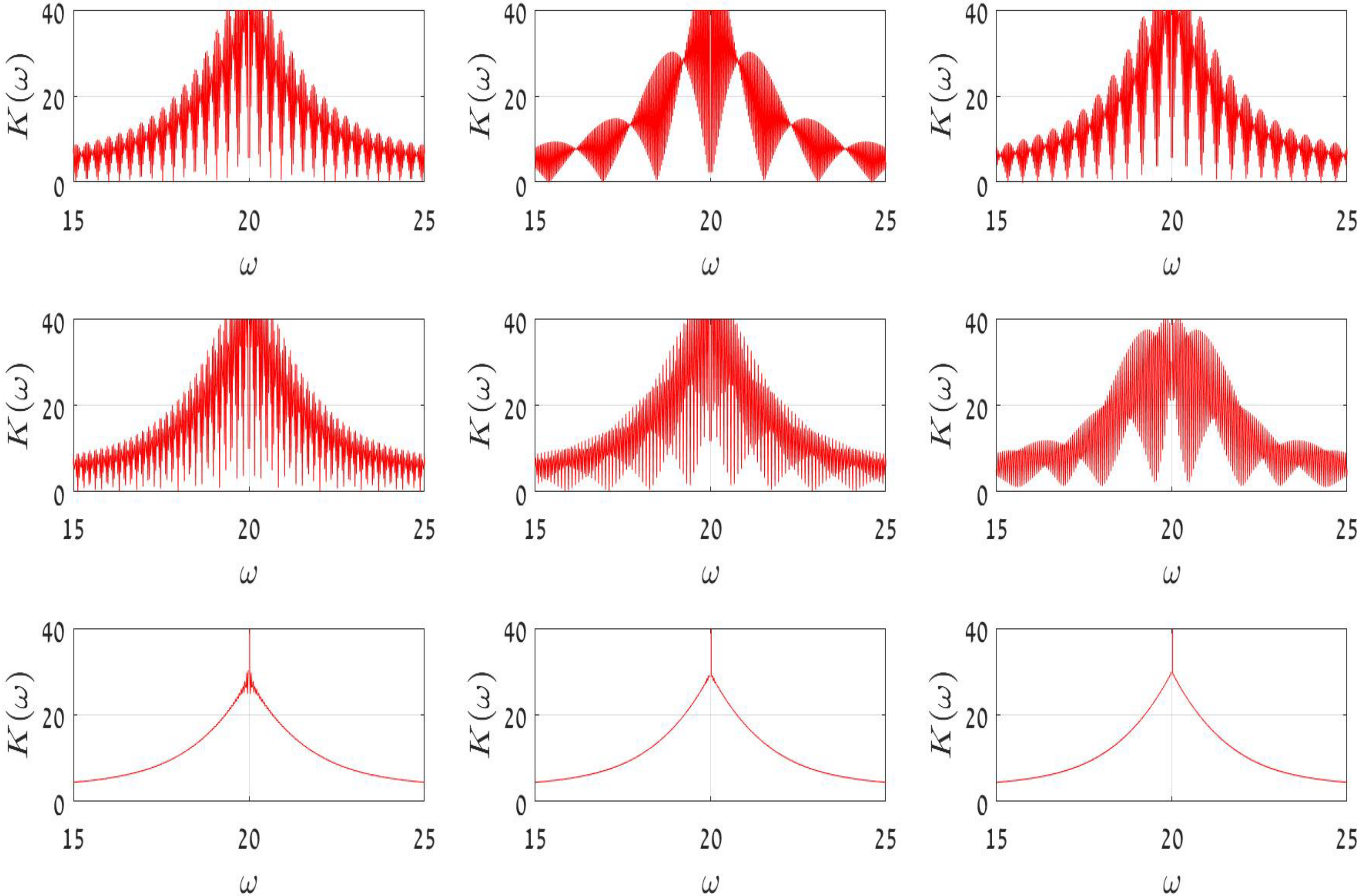
Phase Time Response



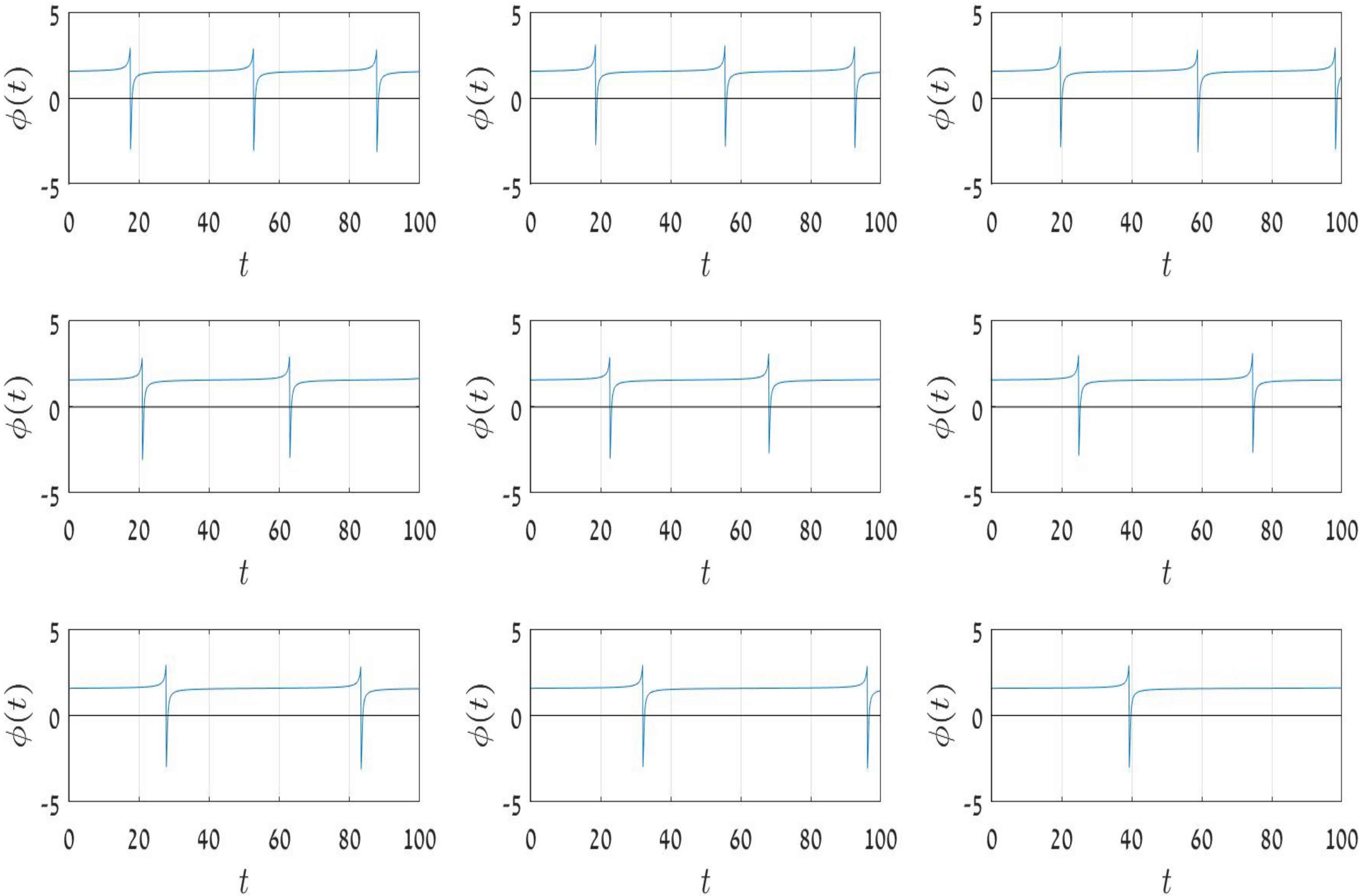
$B = 3$; $K\epsilon[1.0002, 1.001]$

Phase Frequency Response



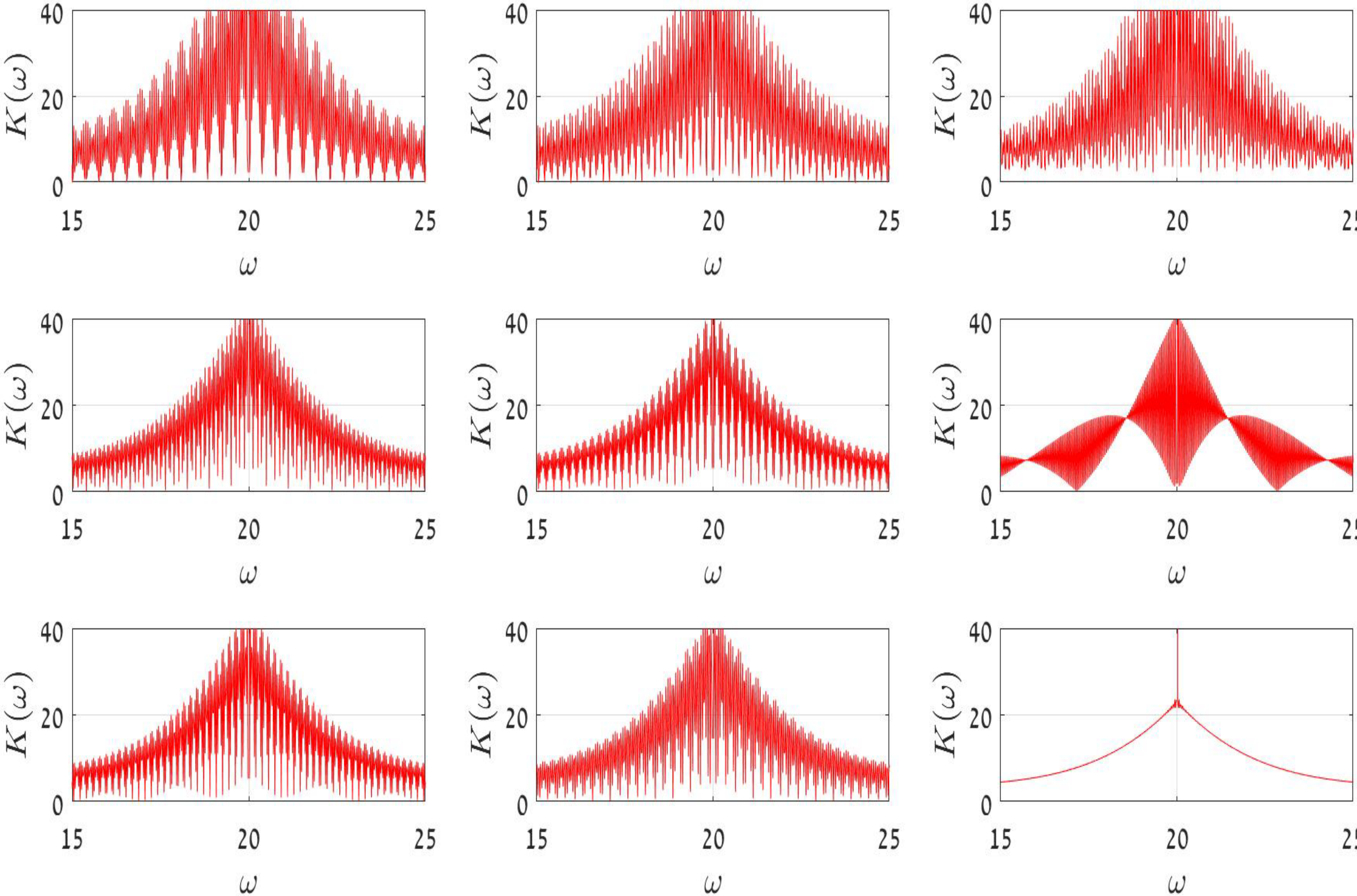
$B = 3$; $K \epsilon [1.0002, 1.001]$

Phase Time Response



$B = 4$; $K\epsilon[1.0002,1.001]$

Phase Frequency Response

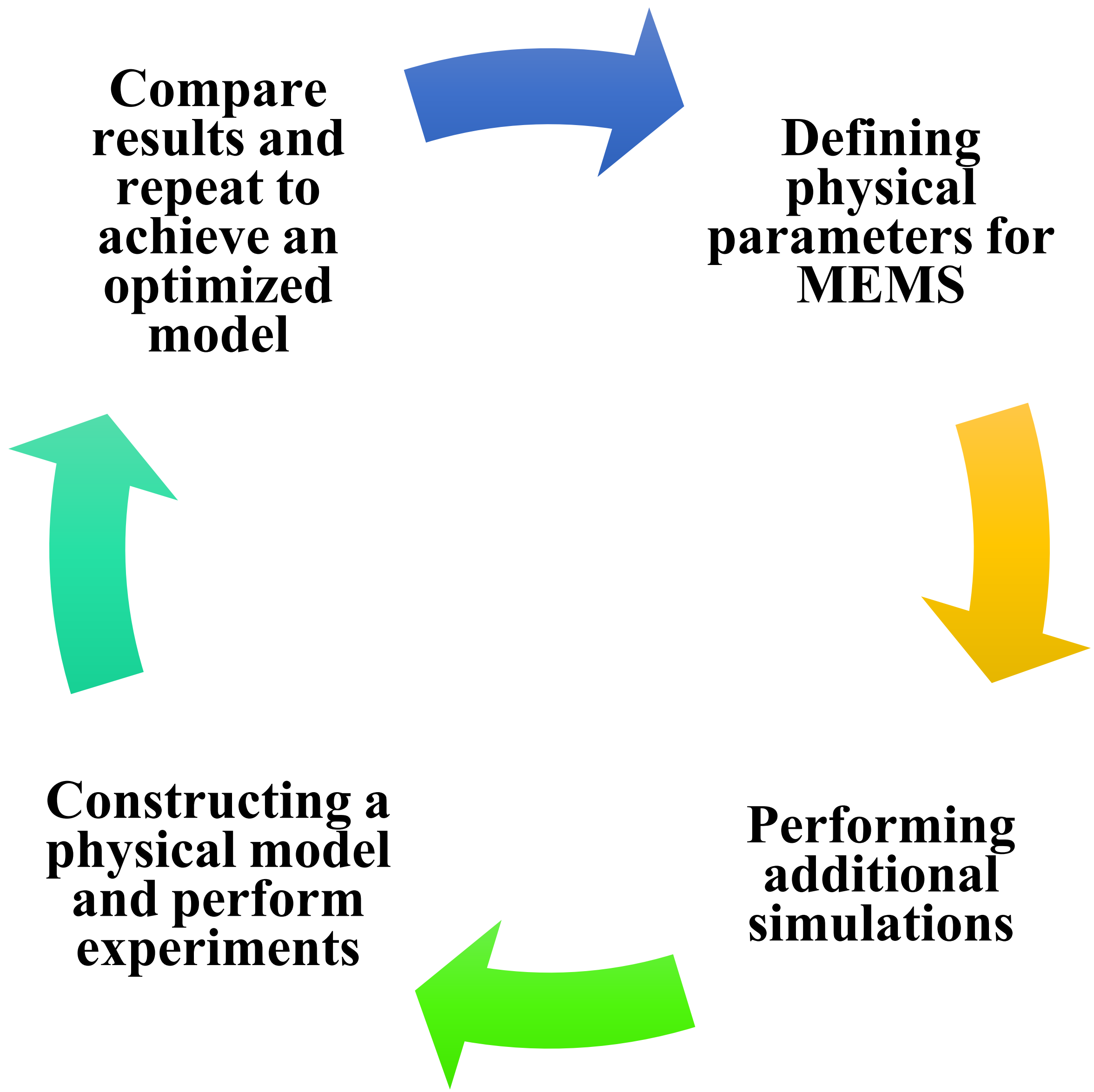


$B = 4$; $K\epsilon[1.0002,1.001]$

Observations and Conclusions

- The control function is satisfactory
- The system dynamics can be reduced to a simple model that can be applicable to MEMS with suiteable parameters
- The model is suiteable for comb extraction
- Values of the parameters highly effect the comb density and overall appearance

Future Work


Defining physical parameters for MEMS
Performing additional simulations
Constructing a physical model and perform experiments
Compare results and repeat to achieve an optimized model