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SNIC bifurcation and its Application to MEMS

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Injection pulling and locking phenomenons occur in specific conditions when two general oscillators interact, in a way that can make them synchronize and oscillate simultaneously to some extant. The idea of harnessing that physical phenomenon in our advantage was covered in great detail and has numerous applications in physics, electrical and mechanical engineering [1] [2]. One interesting application of this interaction is to render a *frequency comb* - a series of discrete, frequency lines in the mathematical frequency domain centered around a *carrier frequency.* Perhaps the most practical application of such a frequency comb is transition from one frequency range to another, e.g., the carrier frequency can be at the extremely high frequency (EHF) band, while the side band of the frequency comb can approach the radio frequency (RF) band (Figure 1).

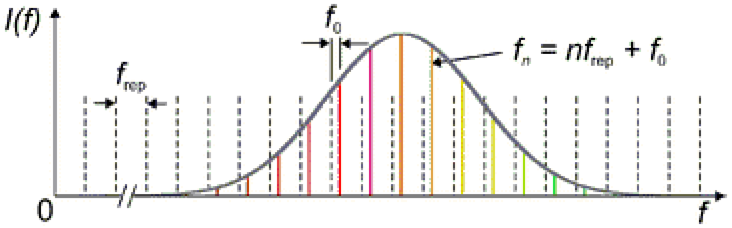

Figure 1: Schematics of a frequency comb—the carrier frequency is at the center of the Lorentzian (orange) and there is a set of discrete frequency lines around it, i.e., a frequency comb.

In this project we discussed the manner in which this phenomenon occurs mechanically, when a mechanical oscillator is excited in such a way that a saddle-node on invariant circle (SNIC) bifurcation occurs and a frequency comb is produced. The implementation of this could relate to many fields, and in this project the relation of the subject to micro electro-mechanical systems (MEMS) was analyzed, hoping to provide some insight into the manner in which a fairly simple mechanical model can generate an output that other, more complex systems produced. The main setback in electrical systems is that they usually involve numerous components that each carry a degree of *noise* in the corresponding output signal they produce, which is often unwanted and usually disrupts the desired outcome and its applications. In that context, a mechanical model can be favourable in that it produces a reduced amount of noise and can be more precise in the results it will provide. The method of execution is the implementation of a closed-loop non-linear microbeam that is perturbed and analyzed for its transverse vibrations (Figure 2).

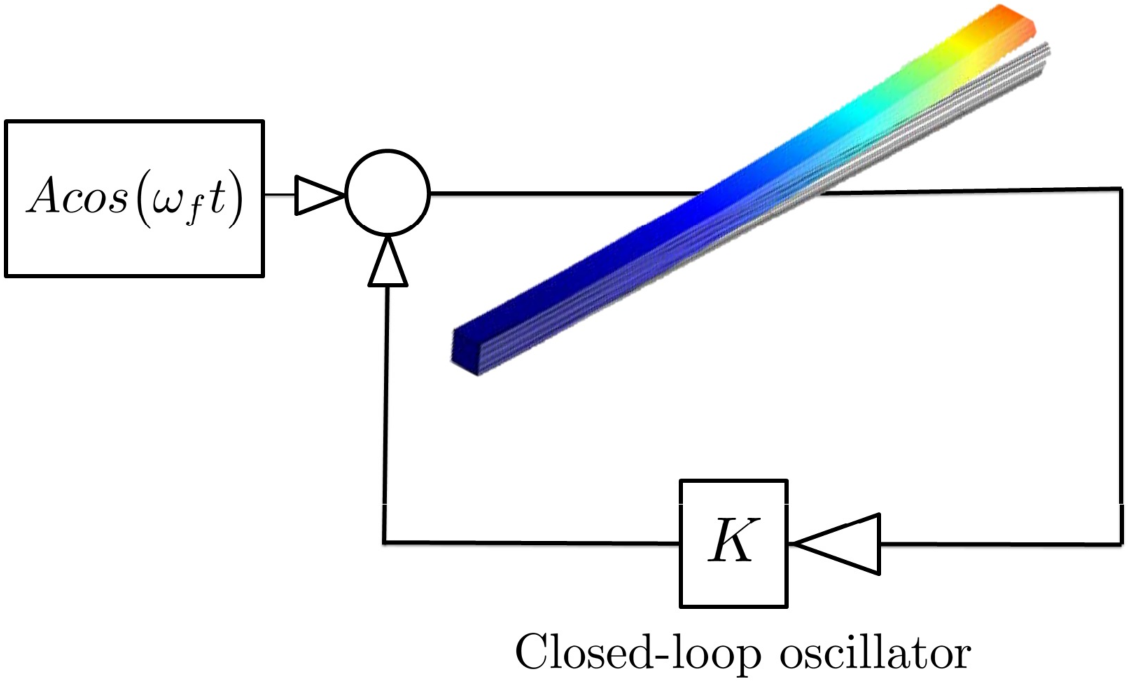

Figure 2: Prototypical perturbed MEMS-based closed-loop oscillator—the micromechanical beam is the frequency selective element, its signal is passing through a phase-shifter and an amplifier, and then fed back to the beam. The adder also add an external perturbation with small amplitude and frequency, which is in the vicinity of the beam eigenfrequency.

The perturbation is an external harmonic driver with a chosen small amplitude and frequency (which is slightly detuned from the beam eigenfrequency), that when engaged with the unperturbed beam oscillations, causes it, due to affects of injection pulling, to reach SNIC bifurcation, rendering a frequency comb as desired.

Theoretical analysis showed that the problem can be modelled using a non-linear equation of the beam, that translates to a form of the non-linear Duffing equation. While a solution to the dynamics function of the beam is hard to obtain in practice due to mathematical difficulties, a slow evolution model is suggested that is composed of a pair of differential equations for the amplitude and the phase. Using several additional mathematical assumptions, the amplitude is seen to be related to the phase, while the phase equation solution is seen to be of the form of Adler's equation [3]. These assumptions ultimately reduce the entire behaviour of the beam to a relatively simple solution of the Adler equation, which has a known analytical solution.

Computerized numerical simulations are run on it to check the results and compare them to the theory and desired outcome. The results agreed with the theory and produce the expected frequency comb, showing the assumptions to be valid in extracting the comb. We solved Adler's equation to get the phase as a function of time and then calculated the solution's *Fourier Transform*, that transforms the function to the complex plane, in order to obtain the gain of the phase as a function of frequency. Using the analytical solution to Adler's equation, one can experiment using the MATLAB® numerical computing environment. Several simulations were made in order to witness the desired behaviour graphically (Figure 3).

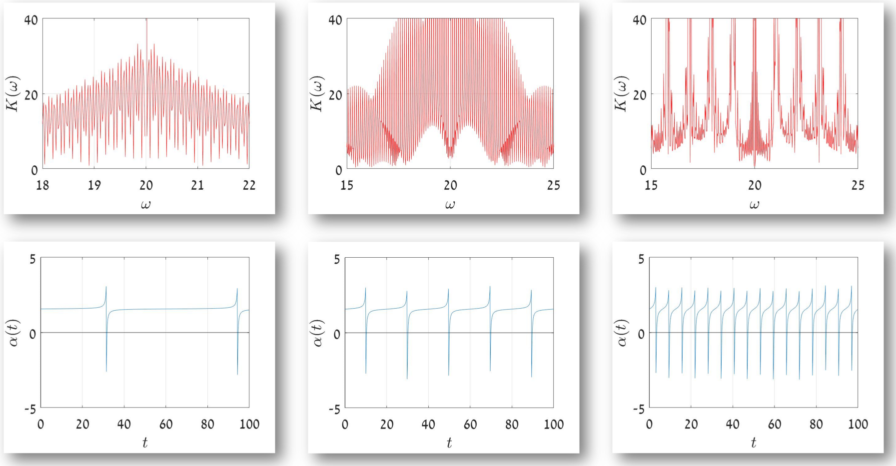

Figure 3: Formation of the frequency comb frequency (upper panels) and time (lower panels) domains— frequency comb for three different parameters in Adler's equation: $K = 2$ (left panel), $K = 1.01$ (center panel), and $K = 1.001$ (right panel)

Observations and Conclusions

- As it shows in the simulation results, as the time signal becomes less dense in smaller set values of iteration, the gain in the frequency domain lowers and one can clearly see a frequency comb forming for the last iterations. These results match our theory, for the values of the penultimate and last iterations are very close to the point of the SNIC bifurcation, in which we predicted that the frequency comb will appear clearly.
- Based on the simulation results, we can conclude that Adler's equation not only fits theoretically as an analytic equation, but is also a fitting model to incite the desired outcome.
- The last iteration displays a very dense comb, one that includes a very large spectrum of frequencies, that is favorable for the applications requiring the comb in the first place.
- Looking at the simulation results, we also see that the time and frequency response of even such a simple model still demonstrate very rapid and sharp changes. Thus, we can conclude that the model is very sensitive to changes in phase, typically near values of bifurcation.

References

[1] Oriel Shoshani, Daniel Heywood, Yushi Yang, Thomas W. Kenny, and Steven W. Shaw. *Phase Noise Reduction in an MEMS Oscillator Using a Nonlinearly Enhanced Synchronization Domain.* Journal of Microelectromechanical Systems, 2016.

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[3] Robert Adler. *A Study of Locking Phenomena in Oscillators.* Proceedings of the IRE, 1946.