Automata and Logic on Infinite Objects 3

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Question 1

(a)

Let $\mathcal{C} = (\Sigma, Q, q_0, \delta, F)$ be an NCW. We are to define an equivalent NPW. Let $\mathcal{P} = (\Sigma, Q, q_0, \delta, \kappa)$ be an NPW with the acceptance condition of: $\mathcal{P}$ is accepting $w \in \Sigma^\omega$ iff there exists a run ρ_w of $\mathcal{P}$ on w such that $\min\{\kappa(inf(\rho_w))\}$ is odd and where:

$$\kappa(q) = \begin{cases} 2 & ; q \in F \\ 3 & ; else \end{cases}$$

(b)

Let $\mathcal{G} = (\Sigma, Q, q_0, \delta, \mathcal{F})$ where $\mathcal{F} = \{F_i\}_{i=1}^k$ be an NGBW. We are to define an equivalent NMW. Let $\mathcal{M} = (\Sigma, Q, q_0, \delta, \alpha)$ be an NMW where:

$$\alpha = \{S \subseteq Q \mid S \cap F_i \neq \emptyset \ ; \ \forall 1 \leq i \leq k\}$$

(c)

Let $\mathcal{S} = (\Sigma, Q, q_0, \delta, \alpha)$ where $\alpha = \{\langle G_i, B_i \rangle\}_{i=1}^k$ be an NSW. We are to define an equivalent NMW. Let $\mathcal{M} = (\Sigma, Q, q_0, \delta, \alpha')$ be an NMW where:

$$\alpha' = \{S \subseteq Q \mid S \cap G_i = \emptyset \vee S \cap B_i \neq \emptyset \ ; \ \forall 1 \leq i \leq k\}$$

(d)

$\mathcal{P} = (\Sigma, Q, q_0, \delta, \kappa)$ be an NPW with $\kappa : Q \to [1, k]$. We are to define an equivalent NSW. Let us define the following sets:

$$\begin{aligned} & I_{even} = \{1 \leq i \leq k \mid i \text{ is even}\} \\ & I_{odd} = \{1 \leq i \leq k \mid i \text{ is odd}\} \\ \forall i \in I_{even} \ ; \quad & (Q_{\text{even}})_i = \{q \mid \kappa(q) = i\} \\ \forall j \in I_{odd} \ ; \quad & (Q_{\text{odd}})_{\leq j} = \{q \mid \kappa(q) = \psi \leq j \wedge \psi \in I_{odd}\} \end{aligned}$$

Let $\mathcal{S} = (\Sigma, Q, q_0, \delta, \alpha)$ where:

$$\alpha = \{\langle (Q_{\text{even}})_i, (Q_{\text{odd}})_{\leq j} \rangle \mid i \in I_{even} \wedge j \in I_{odd}\}$$

Question 2

An alternating 1-Streett automaton (A1SW) is a tuple $\mathcal{A} = (\Sigma, Q, q_0, \delta, \langle G, B \rangle)$ where all the components but the last are as in ABW and $G, B \subseteq Q$. A run-tree r of an A1SW is accepting iff all branches ρ of r satisfy that $\mathit{Inf}(\rho) \cap G \neq \emptyset \to \mathit{Inf}(\rho) \cap B \neq \emptyset$ - which is equivalent to

$$\mathit{Inf}(\rho) \cap G = \emptyset \vee \mathit{Inf}(\rho) \cap B \neq \emptyset$$

We are to provide a construction that converts an A1SW into an equivalent NPW using at most 3 colors.

Construction Idea

Given an A1SW $\mathcal{A} = (\Sigma, Q, q_0, \delta, \langle G, B \rangle)$, we'd want to construct an equivalent NPW $\mathcal{P} = (\Sigma, Q', Q'_0, \delta', \kappa)$. We'll use the min-odd acceptance condition. The construction alters the Miyano-Hayashi construction displayed in class for converting an alternating automaton ABW into an equivalent non-deterministic automaton NBW by eliminating all AND branches.

- **Reminder:** The Miyano-Hayashi construction keeps track of all the paths and makes sure that each path visits an accepting state some time during the run. The construction keeps a booking of all paths that visited an accepting state, and if there comes a time that all paths visit an accepting state - it restarts and once again require all paths to visit an accepting state. This procedure repeats infintely often and if all paths visit an accepting state infintely many times - then the resulting automaton will accept. Thus this construction eliminates AND branches.
- The Miyano-Hayashi construction will not suffice for our problem because in each path we do not know whether we'd want to check that it visits the states in B infintely many times or that it doesn't visit the states in G infintely many times. Thus we'll suggest to guess for each path whether it visits the first acceptance condition: $\mathit{Inf}(\rho) \cap G = \emptyset$ or the second: $\mathit{Inf}(\rho) \cap B \neq \emptyset$.
- $\underline{\mathit{Inf}(\rho) \cap B \neq \emptyset}$: For all paths ρ_j for which we guessed that they would satisfy the condition $\mathit{Inf}(\rho_j) \cap B \neq \emptyset$: we'd look at the resulting run-DAG that the Miyano-Hayashi construction induces on them, but with replacing the original accepting states F with the set B. Let us denote the nodes in a given level i of all of these branches as Q_i. For each i we'll color the nodes in Q_i according to this criterion:
  1. If the level i is not a reset stage in the M-H construction, then we'll color all the nodes in Q_i with the color the 2. This is to signify that at this stage of the run, we haven't still reached a state from B in all of these paths and thus would still want to wait for the time we do. Thus while repeating of these nodes by the min-odd criterion - we'd reject.
  2. If the level i is a reset stage in the M-H construction, then we'll color all the nodes in Q_i with the color the 1. This is to signify (by the min-odd criterion) that we have visited a state from B in all of these paths. Thus while repeating of these nodes (and assuming no other nodes were visited infintely often) by the min-odd criterion - we'd accept.
- $\underline{\mathit{Inf}(\rho) \cap G = \emptyset}$: For all paths ρ_j for which we guessed that they would satisfy the condition $\mathit{Inf}(\rho_j) \cap G = \emptyset$: we'll use a different strategy. Let us consider again the resulting run-DAG that the Miyano-Hayashi construction induces on these paths and let us denote the nodes in a given level i of all of these branches as Q_i. For each i we'll check whether any of the states in Q_i have a state from G, as in if $Q_i \cap G \neq \emptyset$, and color the nodes in Q_i according to this criterion:
  1. If $Q_i \cap G \neq \emptyset$, then we'll color all the nodes in Q_i with the color the 0.
  2. If $Q_i \cap G = \emptyset$, then we'll color all the nodes in Q_i with the color the 1.

Formal Description

Given an A1SW $\mathcal{S} = (\Sigma, Q, q_0, \delta, \langle G, B\rangle)$, the equivalent NPW $\mathcal{P} = (\Sigma, Q', Q'_0, \delta', \kappa)$ is defined as follows: At each initiation of $\mathcal{P}$ on the run-DAG induced by the M-H construction on $\mathcal{S}$ as discussed - each path will be issued with a non-deterministic guess by the accepting conditions mentioned.

- A state of $\mathcal{P}$ will be of the form: $\langle Q_G, Q_{B_v}, Q_{B_o}\rangle$ such that:
  1. Q_G will include all nodes in a given layer such that there exists some path ρ such that their ancestor node was guessing $\mathit{Inf}(\rho) \cap G = \emptyset$.
  2. Q_{B_v} will include all nodes in a given layer such that there exists some path ρ such that their ancestor node was guessing $\mathit{Inf}(\rho) \cap B \neq \emptyset$ and the path ρ has visited a state from B.
  3. Q_{B_o} will include all nodes in a given layer such that there exists some path ρ such that their ancestor node was guessing $\mathit{Inf}(\rho) \cap B \neq \emptyset$ and the path ρ still owes a visit to a state from B.
- Q'_0 will include two states - once corresponding to each guess. Each of them will place the original state q_0 to be in one of the accepting conditions. Thus:

$$Q'_0 = \{\langle q_0, \emptyset, \emptyset\rangle, \langle \emptyset, \emptyset, q_0\rangle\}$$

- Let us define a transition function δ_G for the all paths ρ with the guess $\mathit{Inf}(\rho) \cap G = \emptyset$:

$$\delta_G = \left\{ \Big(\langle Q_G, \emptyset, \emptyset\rangle, \sigma, \langle Q'_G, \emptyset, \emptyset\rangle\Big) \;\middle|\; Q'_G \models \bigwedge_{q \in Q_G} \delta(q, \sigma) \right\}$$

- Let us define a transition function δ_B for the all paths ρ with the guess $\mathit{Inf}(\rho) \cap B \neq \emptyset$

$$\delta_B = \left\{ \Big(\langle \emptyset, Q_{B_v}, Q_{B_o}\rangle, \sigma, \langle \emptyset, Q'_{B_v} \cap B, Q'_{B_v} \setminus B\rangle\Big) \;\middle|\; \begin{array}{l} Q_{B_o} = \emptyset \\ Q'_{B_v} \models \bigwedge_{q \in Q_{B_v}} \delta(q,\sigma) \end{array} \right\} \bigcup$$
$$\left\{ \Big(\langle \emptyset, Q_{B_v}, Q_{B_o}\rangle, \sigma, \langle \emptyset, Q'_{B_v} \cup Q'_{B_o} \setminus \big(Q'_{B_o} \setminus B\big), Q'_{B_o} \setminus B\rangle\Big) \;\middle|\; \begin{array}{l} Q_{B_o} \neq \emptyset \\ Q'_{B_v} \models \bigwedge_{q \in Q_{B_v}} \delta(q,\sigma) \\ Q'_{B_o} \models \bigwedge_{q \in Q_{B_o}} \delta(q,\sigma) \end{array} \right\}$$

  Given these two functions, let us define:

$$\delta'(\langle Q_G, Q_{B_v}, Q_{B_o}\rangle, \sigma) = \bigcup_{\substack{\langle \Psi_G, \emptyset, \emptyset\rangle \in \delta_G \\ \langle \emptyset, \Psi_{B_v}, \Psi_{B_o}\rangle \in \delta_B}} \left\{ \Big(\langle Q'_G, Q'_{B_v}, Q'_{B_o}\rangle\Big) \;\middle|\; \begin{array}{l} Q'_{B_o} \subseteq \Psi_G \cup \Psi_{B_o} \\ Q'_G \uplus Q'_{B_v} \uplus Q'_{B_o} = \Psi_G \cup \Psi_{B_o} \cup \Psi_{B_v} \\ \Psi_G = \emptyset \vee Q'_G \neq \emptyset \\ \Psi_{B_o} \cup \Psi_{B_v} = \emptyset \vee Q'_{B_v} \cup Q'_{B_o} \neq \emptyset \end{array} \right\}$$

  The definition of δ' allows for each node that has ancestors with both accepting condition to "choose" which one he will follow on forth.

- Let us define the coloring as was explained above:

$$\kappa_G(\langle Q_G, \emptyset, \emptyset\rangle) = \begin{cases} 1 & ;\ Q_G \cap G = \emptyset \\ 0 & ;\ else \end{cases}$$

$$\kappa_B(\langle \emptyset, Q_{B_v}, Q_{B_o}\rangle) = \begin{cases} 1 & ;\ Q_{B_o} = \emptyset \\ 2 & ;\ else \end{cases}$$

$$\kappa(\langle Q_G, Q_{B_v}, Q_{B_o}\rangle) = \begin{cases} \kappa_B(\langle \emptyset, Q_{B_v}, Q_{B_o}\rangle) & ;\ \kappa_G(\langle Q_G, \emptyset, \emptyset\rangle) = 1 \\ 0 & ;\ else \end{cases}$$

Construction Proof

Claim 0.1. *Let $w \in \Sigma^\omega$. Let $\mathcal{S}$ be an A1SW. Then the NPW $\mathcal{P}$ by the construction above accepts w iff $\mathcal{S}$ accepts w.*

Proof. Let $r = \langle Q_G^0, Q_{B_v}^0, Q_{B_o}^0 \rangle, \langle Q_G^1, Q_{B_v}^1, Q_{B_o}^1 \rangle \ldots$ be an accepting run of $\mathcal{P}$ on $w \in \Sigma^\omega$. Hence, since there are only three colors $\{0, 1, 2\}$ $min(k(inf(r))) = 1$ (acceptance condition is defined to be min, odd). By definition of k, $(min(k_G(inf(r))) = 1) \wedge (min(k_B(inf(r))) = 1)$. By the Acceptance of an ω-word by the A1SW. A run-tree r of an A1SW is accepting iff all branches p of r satisfy that $Inf(p) \cap G \neq \emptyset$ implies $Inf(p) \cap B \neq \emptyset$. This condition is equivalent to $Inf(p) \cap G = \emptyset \vee Inf(p) \cap B \neq \emptyset$. Let ρ be a branch in r, split into cases:
∘ We guess the first accepting option, that is Inf(ρ) $\cap B \neq \emptyset$, in that case, our construction is just like the M-H construction we saw in class. Thus from the correctness of M-H, we get that there has been infinitely reset steps, we know that between every two adjacent reset points all paths have visited at $b \in B$ state at least once, and therefore all paths have visited the accepting set infinitely often. Thus ρ satisfies Inf(ρ) $\cap B \neq \emptyset$.
∘ We guess the second accepting option, that is Inf(ρ) $\cap G = \emptyset$, in that case, $min(k_G(inf(\rho))) = 1$, by definition of k_G, ρ visited only finite times in $q_i \in G$, otherwise the minimal color would be zero and even in contradiction to r an accepting run on $\mathcal{P}$.
In conclusion, if a word $w \in \mathcal{P}$ then $w \in \mathcal{S}$.

Now, let us consider $\langle T, v \rangle$ an accepting run-DAG of $\mathcal{S}$ on w. Thus by its definition, each of his branches ρ satisfies *Inf*(ρ) $\cap G = \emptyset \vee$ *Inf*(ρ) $\cap B \neq \emptyset$. Thus by the definition of κ we get that the whole run ρ^* satisfies min($\kappa(Inf(\rho^*))$) = 1. Let us now split into cases for each sub-branch ρ in the run ρ^*:

- If ρ satisfies *Inf*(ρ) $\cap B \neq \emptyset$ then since we have infintely many resets by our construction - we'll have by definiton $\kappa_G = \kappa_B = 1$ and - then $\kappa = 1$ and thus the path is accepted.
- If ρ satisfies $Inf(p) \cap G = \emptyset =$ then there exists a point in ρ^* in which we stop seeing states from G. Let us denote the nodes in a given level i as Q_i and let ψ be the index of the last visit to a state from G. Then we must have $\forall n > \psi$; $Q_n \cap G = \emptyset$. Thus by the definition of κ we'll have $\kappa = 1$ and the path is accepted.
- If there exists a path which is not accepted - we'll have that both condition are not satisfied and thus we'll have $min(k_G(inf(\rho))) = 0$ and thus we do not accept.

□

Question 3

(a)

In this question we are to prove the Robustness of the Weak Wagner Classes:

Theorem 0.2. *Let $L = [\![\mathcal{M}]\!] \in \mathbb{DM}_{\mathbb{1}}^{\pm}$ for a DMW $\mathcal{M}$ such that $|\mathcal{M}|_{\rightsquigarrow}^{+} = d$ and $|\mathcal{M}|_{\rightsquigarrow}^{-} = d'$. Let $\mathcal{M}'$ be a different DMW such that $[\![\mathcal{M}']\!] = L$. Then $|\mathcal{M}'|_{\rightsquigarrow}^{+} = d$ and $|\mathcal{M}'|_{\rightsquigarrow}^{-} = d'$.*

Proof. We will prove the theorem for $|\mathcal{M}|_{\rightsquigarrow}^{+}$ since the same argument for $|\mathcal{M}|_{\rightsquigarrow}^{-}$ is symmetrical. Let $\mathcal{M}$ be a DMW such that $[\![\mathcal{M}]\!] \in \mathbb{DM}_{\mathbb{1}}^{\pm}$. Let us denote $[\![\mathcal{M}]\!] = L$ and let us assume that $|\mathcal{M}|_{\rightsquigarrow}^{+} = d$. We will prove the theorem for an odd value of d since the same argument for an even value of d is similar up to a change of indices. Since $|\mathcal{M}|_{\rightsquigarrow}^{+} = d$, then there exists a sequence of maximal strongly connected components (MSCCs) in $\mathcal{M}$ of the form: $S_1 \rightsquigarrow S_2 \rightsquigarrow \cdots \rightsquigarrow S_d$, with alternating polarities such that S_i is reachable from S_{i-1} for $i \leq d$ and S_1 is an accepting MSCC. For each $1 \leq i \leq d$ let s_i be a state of S_i and let ψ_i be a word that takes s_i back to s_i while visiting all of S_i's states and no other states. Moreover, for each $1 \leq j \leq d-1$ let ζ_j be a word that takes ψ_j to ψ_{j+1}. Let Ψ be a word reaching s_1. Let $\mathcal{M}'$ be a different DMW such that $[\![\mathcal{M}']\!] = L$ and let us denote the number of states in $\mathcal{M}'$ as n. Let $m > n$ and let us consider the following ω-words:

$$\begin{aligned}
w_1 = &\quad \Psi \cdot \psi_1^{\omega} \\
w_2 = &\quad \Psi \cdot \psi_1^{m} \cdot \zeta_1 \cdot \psi_2^{\omega} \\
w_3 = &\quad \Psi \cdot \psi_1^{m} \cdot \zeta_1 \cdot \psi_2^{m} \cdot \zeta_2 \cdot \psi_3^{\omega} \\
&\quad \vdots \\
w_d = &\quad \Psi \cdot \psi_1^{m} \cdot \zeta_1 \cdot \psi_2^{m} \cdot \zeta_2 \cdot \psi_3^{m} \cdot \zeta_3 \cdots \zeta_{d-1} \cdot \psi_d^{\omega}
\end{aligned}$$

By the definition of the positive diameter measure $|\mathcal{M}|_{\rightsquigarrow}^{+}$ - one can see that for every $1 \leq i \leq d$ we have that $w_i \in L$ iff i is odd.

Lemma 0.3. $|\mathcal{M}'|_{\rightsquigarrow}^{+} \geq d$

Proof. From the previous statement and because we assumed that d is odd - we have that $w_d \in L = [\![\mathcal{M}']\!]$. Let S'_d be the trapping MSCC of $\mathcal{M}'$ on w_d. Since $w_d \in [\![\mathcal{M}']\!]$ we have that S'_d is an accepting MSCC of $\mathcal{M}'$. Let us denote ρ_d to be the run of $\mathcal{M}'$ on w_d (which is singular since $\mathcal{M}'$ is deterministic). Since $m > n$, and n is the number of states in $\mathcal{M}'$, we have that ρ_d must have a loop when going through the infix $\psi_1^{m} \cdot \zeta_1 \cdot \psi_2^{m} \cdot \zeta_2 \cdot \psi_3^{m} \cdot \zeta_3 \cdots \psi_{d-1}^{m}$. This loop must form an MSCC which we will denote S'_{d-1}, which must go on with ζ_{d-1} to S'_d. Thus $S'_{d-1} \rightsquigarrow S'_d$. Continuing in the same manner we obtain a sequence of reachable MSCCs of $\mathcal{M}'$ of the form: $S'_1 \rightsquigarrow S'_2 \rightsquigarrow \cdots \rightsquigarrow S'_{d-1} \rightsquigarrow S'_d$ with alternating polarities S'_1 is accepting. Thus, by definition, we have that $\mathcal{M}'$ has a positive diameter measure of at least d. □

Lemma 0.4. $|\mathcal{M}'|_{\rightsquigarrow}^{+} \leq d$

Proof. Let us assume towards contradiction that $\mathcal{M}'$ has a positive diameter measure of more than d, as in $|\mathcal{M}'|_{\rightsquigarrow}^{+} = r > d$. Applying the same argument stated in the previous lemma with reversed roles between $\mathcal{M}$ and $\mathcal{M}'$ we'll get that $\mathcal{M}$ must have a positive diameter measure of at least r, as in $|\mathcal{M}|_{\rightsquigarrow}^{+} \geq r > d$, in contradiction to the assumption that $|\mathcal{M}|_{\rightsquigarrow}^{+} = d$. □

From these two lemmas we'll get $|\mathcal{M}'|_{\rightsquigarrow}^{+} = d$. □

(b)

In this question we are to prove the Strictness of the Weak Wagner Classes:

Theorem 0.5. *1.* $\mathbb{DM}_{\mathbb{1}}^{(d,\pm)} \subsetneq \mathbb{DM}_{\mathbb{1}}^{(d+1,p)}$ *for* $p \in \{+,-\}$ *and* $d \in \mathbb{N}$.

Proof. Let $p \in \{+,-\}$ and $d \in \mathbb{N}$. By definition we have that:

$$\begin{aligned}
\mathbb{DM}_{\mathbb{1}}^{(d,\pm)} &= \{L \mid \exists \text{ DMA } \mathcal{M} \in \mathbb{DM}_{\mathbb{1}}^{\pm} \text{ s.t. } |\mathcal{M}|_{\rightsquigarrow}^{+} \leq d \ \wedge \ |\mathcal{M}|_{\rightsquigarrow}^{-} \leq d\} \\
&= \{L \mid \exists \text{ DMA } \mathcal{M} \in \mathbb{DM}_{\mathbb{1}}^{\pm} \text{ s.t. } |\mathcal{M}|_{\rightsquigarrow}^{+} < d+1 \ \wedge \ |\mathcal{M}'|_{\rightsquigarrow}^{-} < d+1\} \\
\mathbb{DM}_{\mathbb{1}}^{(d+1,p)} &= \{L \mid \exists \text{ DMA } \mathcal{M} \in \mathbb{DM}_{\mathbb{1}}^{\pm} \text{ s.t. } |\mathcal{M}|_{\rightsquigarrow}^{p} \leq d+1\}
\end{aligned}$$

Thus we trivially have that $\mathbb{DM}_{\mathbb{1}}^{(d,\pm)} \subseteq \mathbb{DM}_{\mathbb{1}}^{(d+1,p)}$.

Lemma 0.6. *Let* $p \in \{+,-\}$ *and* $d \in \mathbb{N}$. *Then* $\mathbb{DM}_{\mathbb{1}}^{(d,\pm)} \neq \mathbb{DM}_{\mathbb{1}}^{(d+1,p)}$.

Proof. Let $d \in \mathbb{N}$ and let us assume w.l.o.g that $p = +$. The argument for $p = -$ is symmetrical. We will prove the lemma by providing a DMW $\mathcal{M}$ such that $\mathcal{M} \in \mathbb{DM}_{\mathbb{1}}^{(d+1,+)} \setminus \mathbb{DM}_{\mathbb{1}}^{(d,\pm)}$. Let $\mathcal{M} = (\Sigma, Q, q_0, \delta, \alpha)$ be a DMW where:

$$\begin{aligned}
&\Sigma = \{a, b\} \\
&Q = \{q_i \mid 1 \leq i \leq d+1\} \\
&q_0 = q_1 \\
&\delta(q_i, b) = q_i \\
&\delta(q_i, a) = q_{i+1} \ ; \ \forall 1 \leq i \leq d \\
&\delta(q_{d+1}, a) = q_{d+1} \\
&\alpha = \{\{q_i\} \mid i \text{ is odd}\}
\end{aligned}$$

One can see that $\mathcal{M} \in \mathbb{DM}_{\mathbb{1}}^{\pm}$, since $\mathcal{M}$ has no MSCC with a subsumed SCC with different polarity. Moreover, $\mathcal{M}$ has a positive diameter measure of $d+1$ and a negative diameter measure of d so $|\mathcal{M}|_{\rightsquigarrow}^{+} = d+1$ and $|\mathcal{M}|_{\rightsquigarrow}^{-} = d$. Thus we have by definition that $\mathcal{M} \in \mathbb{DM}_{\mathbb{1}}^{(d+1,+)}$ and since $d < d+1 = |\mathcal{M}|_{\rightsquigarrow}^{+}$ we have also by definition that $\mathcal{M} \notin \mathbb{DM}_{\mathbb{1}}^{(d,+)}$. Since $\mathbb{DM}_{\mathbb{1}}^{(d,+)} \subseteq \mathbb{DM}_{\mathbb{1}}^{(d,\pm)}$ - we get that $\mathcal{M} \notin \mathbb{DM}_{\mathbb{1}}^{(d,\pm)}$. □

Since $\mathbb{DM}_{\mathbb{1}}^{(d,\pm)} \subseteq \mathbb{DM}_{\mathbb{1}}^{(d+1,p)}$ and by the lemma we have that $\mathbb{DM}_{\mathbb{1}}^{(d,\pm)} \neq \mathbb{DM}_{\mathbb{1}}^{(d+1,p)}$ - we have that $\mathbb{DM}_{\mathbb{1}}^{(d,\pm)} \subsetneq \mathbb{DM}_{\mathbb{1}}^{(d+1,p)}$. □

Theorem 0.7. *2.* $\mathbb{DM}_1^{(d,p)} \subsetneq \mathbb{DM}_1^{(d,\pm)}$ *for* $p \in \{+,-\}$ *and* $d \in \mathbb{N}$.

Proof. Let $p \in \{+,-\}$ and $d \in \mathbb{N}$. First, by arguments similar to those in the previous section, we trivially have that $\mathbb{DM}_{\mathbb{1}}^{(d,p)} \subseteq \mathbb{DM}_{\mathbb{1}}^{(d,\pm)}$ by definition.

Lemma 0.8. *Let* $p \in \{+,-\}$ *and* $d \in \mathbb{N}$. *Then* $\mathbb{DM}_{\mathbb{1}}^{(d,p)} \neq \mathbb{DM}_{\mathbb{1}}^{(d,\pm)}$.

Proof. Let $d \in \mathbb{N}$ and let us assume w.l.o.g that $p = +$. The argument for $p = -$ is symmetrical. We will prove the lemma by providing a DMW $\mathcal{M}$ such that $\mathcal{M} \in \mathbb{DM}_{\mathbb{1}}^{(d,\pm)} \setminus \mathbb{DM}_{\mathbb{1}}^{(d,+)}$. Let $\mathcal{M} = (\Sigma, Q, q_0, \delta, \alpha)$ be a DMW where:

$$\begin{aligned}
&\Sigma = \{a, b\} \\
&Q = \{q_i \mid 1 \leq i \leq d\} \\
&q_0 = q_1 \\
&\delta(q_i, b) = q_i \\
&\delta(q_i, a) = q_{i+1} \ ; \ \forall 1 \leq i \leq d-1 \\
&\delta(q_d, a) = q_d \\
&\alpha = \{\{q_i\} \mid i \text{ is even}\}
\end{aligned}$$

One can see that $\mathcal{M} \in \mathbb{DM}_{\mathbb{1}}^{\pm}$, since $\mathcal{M}$ has no MSCC with a subsumed SCC with different polarity. Moreover, $\mathcal{M}$ has a positive diameter measure of $d-1$ and a negative diameter measure of d so $|\mathcal{M}|_{\rightsquigarrow}^{+} = d-1$ and $|\mathcal{M}|_{\rightsquigarrow}^{-} = d$. Thus we have by definition that $\mathcal{M} \in \mathbb{DM}_{\mathbb{1}}^{(d,-)}$ and $\mathcal{M} \notin \mathbb{DM}_{\mathbb{1}}^{(d,+)}$. Since $\mathbb{DM}_{\mathbb{1}}^{(d,-)} \subseteq \mathbb{DM}_{\mathbb{1}}^{(d,\pm)}$ - we get that $\mathcal{M} \in \mathbb{DM}_{\mathbb{1}}^{(d,\pm)}$. □

Since $\mathbb{DM}_{\mathbb{1}}^{(d,p)} \subseteq \mathbb{DM}_{\mathbb{1}}^{(d,\pm)}$ and by the lemma we have that $\mathbb{DM}_{\mathbb{1}}^{(d,p)} \neq \mathbb{DM}_{\mathbb{1}}^{(d,\pm)}$ - we have that $\mathbb{DM}_1^{(d,p)} \subsetneq \mathbb{DM}_1^{(d,\pm)}$. □

Question 4

In this question we are to prove the following:

Theorem 0.9. $\mathbb{NBT} \supsetneq \mathbb{DBT}$

Proof. Let:

$$L_1 = \{\Sigma\text{-labeled } D\text{-trees } \langle T, v\rangle \mid \exists \text{ path } \pi \in T \text{ s.t. } a \text{ occurs only finitely many times in } \pi\}$$

where:

$$\Sigma = \{a, b\}$$
$$D = \{0, 1\}$$

We will provide an NBT accepting L_1. Let $\mathcal{T} = (\Sigma, Q, Q_0, \delta, F)$ where:

$$Q = \{q_{F(a)}, q_{G(b)}, q_{acc}, q_{rej}\}$$
$$Q_0 = \{q_{F(a)}\}$$
$$\delta(q_{F(a)}, a) = \delta(q_{F(a)}, b) = \{\langle q_{F(a)}, q_{acc}\rangle, \langle q_{acc}, q_{F(a)}\rangle, \langle q_{G(b)}, q_{acc}\rangle, \langle q_{acc}, q_{G(b)}\rangle\}$$
$$\delta(q_{G(b)}, b) = \{\langle q_{G(b)}, q_{acc}\rangle, \langle q_{acc}, q_{G(b)}\rangle\}$$
$$\delta(q_{G(b)}, a) = \{\langle q_{rej}, q_{rej}\rangle\}$$
$$\delta(q_{acc}, a) = \delta(q_{acc}, b) = \{\langle q_{acc}, q_{acc}\rangle\}$$
$$\delta(q_{rej}, a) = \delta(q_{rej}, b) = \{\langle q_{rej}, q_{rej}\rangle\}$$
$$F = \{q_{G(b)}, q_{acc}\}$$

Let us provide a short argument as to why $\mathcal{T}$ accepts L_1:

- $\mathcal{T}$ works by guessing the point at which the letter a will stop occurring in some path.
- The state $q_{F(a)}$ symbols that we still expect to see another a somewhere down the current path (and thus the notation $F(a)$ for *"finally a"*).
- At some point, $\mathcal{T}$ guesses that from now on it will only encounter b's along the current path - a condition assigned to the state $q_{G(b)}$ (and thus the notation $G(b)$ for *"globally b"*).
- The run of $\mathcal{T}$ starts off with accepting any letter and allowing at each point to be the designated location of the guess, while accepting the other side (using the designated state q_{acc}), or keep on seeing a's, and accepting on the other side.
- When the guess finally occurs, we'd want to keep on seeing b's - so with a b we'll keep looking for that condition using the state $q_{G(b)}$ and accepting at the other side. If from any point after the guess we see an a - we'll reject - as denoted by the state q_{rej}.
- $\mathcal{T}$ would accept a Σ-labeled D-tree $\langle T, v\rangle$ that has $\langle T, v\rangle \in L_1$ because T has a path π such that a occurs only finitely many times in π, and thus there exists a run of $\mathcal{T}$ on $\langle T, v\rangle$ in which we will stop seeing a's along a certain path - so the paths down that line will accept, along with the other paths that will accept as explained before.
- $\mathcal{T}$ would reject a Σ-labeled D-tree $\langle T, v\rangle$ that has $\langle T, v\rangle \notin L_1$ because T will have a's appearing infinitely many times in each of its paths so any guess of $\mathcal{T}$ will be wrong - as in there exists a path π in all runs of $\mathcal{T}$ on $\langle T, v\rangle$ (the one of the guess) in which we would only reject infinitely many times - and thus won't see any accepting states infinitely many times - and then we'll have $\mathit{Inf}(\pi) \cap F = \emptyset$.

Claim 0.10. *There is no DBT accepting L_1.*

Proof. Assume towards contradiction that exits a DBT $\mathcal{D} = (\Sigma, Q, q_0, \delta, F)$ accepting L_1. Let us denote $|F| = \Psi$. Let $\langle T, v\rangle$ be a tree such that all of its paths contain infinitely many a's but for one - π^* - in which there are only b's, as in the path π^* corresponds to b^ω. Clearly, $\langle T, v\rangle \in L_1$. Since we assumed that $\mathcal{D}$ accepts L_1 we have that $[\![\mathcal{D}]\!] = L_1$ and thus $\langle T, v\rangle \in [\![\mathcal{D}]\!]$. Thus the run $\langle T, r\rangle$ of $\mathcal{D}$ on $\langle T, v\rangle$ (which is singular since $\mathcal{D}$ is deterministic) is accepting - which means that all paths $\pi \in T$ have $Inf(\pi) \cap F = \emptyset$, and in particular we have that $Inf(\pi^*) \cap F = \emptyset$ - as in the path π visits some accepting state infintely often. Let ψ_i be the index in π^* in which we visit an accepting state the i'th time. Let us consider a set of modified trees $\{\langle T_i, v\rangle\}_{i=1}^{\Psi+1}$ in which T_i is T but with π^* replaced with π_i^* - the path corresponding to $b^{\psi_1}ab^{\psi_2}a \cdots ab^{\psi_i}ab^\omega$. For any $1 \leq i \leq \Psi$, there is a finite number of a's in π_i^* so we have $\langle T_i, v\rangle \in [\![\mathcal{D}]\!]$. Since $\mathcal{D}$ is deterministic, the runs $\langle T, r\rangle$ of the original tree and $\langle T_i, r_i\rangle$ of the modified trees agree at each stage before the change of π^* at ψ_i. The tree $\langle T_{\Psi+1}, v\rangle$ will have a path $\pi^*_{\Psi+1}$ that corresponds to $w = b^{\psi_1}ab^{\psi_2}a \cdots ab^{\psi_{\Psi+1}}ab^\omega$. By our construction - the run $\langle T, r\rangle$ of $\mathcal{D}$ on $\langle T, v\rangle$ visits an accepting state in each index a appears in w. Since there are $\Psi + 1$ a's in w, there are $\Psi + 1$ visits of an accepting state. Since we assumed that $|F| = \Psi$, by the Pigeonhole Principle we have that there must be two indices $1 \leq r < s \leq \Psi + 1$ such that $q_{\psi_r} = q_{\psi_s}$ when q_{ψ_m} represents the accepting state visited at the m'th iteration of the construction. Let us consider $\langle T^\#, v\rangle$ to be a tree where $T^\#$ is T but with π^* replaced with the path $\pi^\#$ corresponding to: $b^{\psi_1}ab^{\psi_2}a \cdots ab^{\psi_r}(ab^{\psi_{r+1}}ab^{\psi_{r+2}}a \cdots ab^{\psi_s})^\omega$. Since there are infinitely many a's now in $\pi^\#$ there is no path where a occurs only a finite number of times and thus we have that $\langle T^\#, v\rangle \notin L_1$. But since the run $\langle T^\#, r^\#\rangle$ of $\mathcal{D}$ on $\langle T^\#, v\rangle$ is deterministic, it still agrees with $\langle T, r\rangle$ on each visit to an accepting state - and thus it visits some accepting state infintely often and thus we have that $\langle T^\#, v\rangle \in [\![\mathcal{D}]\!]$, in contradiction to the assumption that $[\![\mathcal{D}]\!] = L_1$.

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