Automata and Logic on Infinite Objects 1

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1 Question 1

Let r be an ω-regular expression and let Σ be an alphabet such that $r \in \Sigma \cup \{\emptyset, \cdot, ^{\omega}, +\}$. We'll show that there exists an NBW $\mathcal{B}_r$ such that $[\![\mathcal{B}_r]\!] = [\![r]\!]$.
Reminder: Let us recall that an NBW is a tuple $\mathcal{B}_r = (\Sigma, Q, Q_0, \Delta, F)$. For a run $\rho = q_0 q_1 q_2 ...$ let us define $inf(\rho) = \{q \in Q \mid \forall i \in \mathbb{N} j > i q_j = q\}$ - the set of states visited infinitely often during the run ρ. The Büchi acceptance condition is the set $F \subseteq Q$ and a run ρ of a Büchi automaton is accepting if it visits F infinitely often, as in if $inf(\rho) \cap F \neq \emptyset$.
We'll use complete structural induction on $|r|$ - the length of r.

- Base case: Since r is a ω-regular expression, $|r| > 0$. Therefore, the base case will be for $|r| = 1$. In that case, by the definition of ω-regular expressions, it must be that $r = \emptyset$. In that case, by definition, $[\![r]\!] = [\![\emptyset]\!] = \emptyset$. Let $\mathcal{B}_r$ be an NBW with one non-accepting state.
  Formally: $\mathcal{B}_r = (\Sigma, Q, Q_0, \Delta, F)$ such that:

$$Q = \{q\}$$
$$Q_0 = \{q\}$$
$$\forall \sigma \in \Sigma \; ; \; \Delta(\sigma, q) = \{q\}$$
$$F_r = \emptyset$$

  According to the Büchi acceptance condition - for any run ρ it will hold that: $inf(\rho) \cap F = inf(\rho) \cap \emptyset = \emptyset$ and therefore $[\![\mathcal{B}_r]\!] = \emptyset = [\![r]\!]$.

- Induction assumption: Let r be a ω-regular expression such that $1 < |r| < n$. So there exists an NBW $\mathcal{B}_r$ such that $[\![\mathcal{B}_r]\!] = [\![r]\!]$.

- Induction step: Let r be a ω-regular expression such that $|r| = n > 1$. Since $|r| > 1$, there exists two ω-regular expression r_1, r_2 such that one of the following holds:

  1. $r = r_1 + r_2$ where r_1 and r_2 are ω-regular expressions.
  2. $r = r_1 \cdot r_2$ where r_1 is a regular expression and r_2 is an ω-regular expressions.
  3. $r = r_1^{\omega}$ where r_1 is a regular expression.

  In all these cases, it holds that $|r_1| < n$ and $|r_2| < n$ and so the induction assumption holds for r_1 and r_2. Let us denote $[\![r_1]\!] = L_1$ and $[\![r_1]\!] = L_2$.
  Let us now split into the 3 aforementioned cases:

  1. $r = r_1 + r_2$:
     From the induction assumption we'll get that there exist two NBWs $\mathcal{B}_{r_1} = (\Sigma, Q_{r_1}, Q_{0_{r_1}}, \Delta_{r_1}, F_{r_1})$ and $\mathcal{B}_{r_2} == (\Sigma, Q_{r_2}, Q_{0_{r_2}}, \Delta_{r_2}, F_{r_2})$ such that $[\![\mathcal{B}_{r_1}]\!] = [\![r_1]\!] = L_1$ and $[\![\mathcal{B}_{r_2}]\!] = [\![r_2]\!] = L_2$.
     Applying the semantics function on both sides of the equation yields:

$$[\![r]\!] = [\![r_1 + r_2]\!] = [\![r_1]\!] \cup [\![r_2]\!] = [\![\mathcal{B}_{r_1}]\!] \cup [\![\mathcal{B}_{r_2}]\!] = L_1 \cup L_2$$

We spoke in class of a construction for an NBW that accepts a union of two NBWs so we will provide a short correctness argument: let $\mathcal{B}_r = (\Sigma, Q_r, Q_{0_r}, \Delta_r, F_r)$ be a NBA such that:

$$Q_r = Q_{r_1} \cup Q_{r_2}$$
$$Q_{0_r} = Q_{0_{r_1}} \cup Q_{0_{r_2}}$$
$$\Delta_r = \Delta_{r_1} \cup \Delta_{r_2}$$
$$F_r = F_{r_1} \cup F_{r_2}$$

$\mathcal{B}_r$ starts with all the accepting states of $\mathcal{B}_{r_1}$ and $\mathcal{B}_{r_2}$, transitions and accepts according to them - so it accepts the language that is the union $L_1 \cup L_2$. So it will hold that:

$$[\![\mathcal{B}_r]\!] = L_1 \cup L_2 = [\![r]\!]$$

2. $r = r_1 \cdot r_2$:
From the induction assumption we'll get that: since r_1 is a regular expression, there exists an NFW $\mathcal{N}_{r_1} = (\Sigma, Q_{r_1}, Q_{0_{r_1}}, \Delta_{r_1}, F_{r_1})$ such that $[\![\mathcal{N}_{r_1}]\!] = [\![r_1]\!] = L_1$ and since r_2 is an ω-regular expression, there exists an NBW $\mathcal{B}_{r_2} = (\Sigma, Q_{r_2}, Q_{0_{r_2}}, \Delta_{r_2}, F_{r_2})$ such that $[\![\mathcal{B}_{r_2}]\!] = [\![r_2]\!] = L_2$. Applying the semantics function on both sides of the equation yields:

$$[\![r]\!] = [\![r_1 \cdot r_2]\!] = [\![r_1]\!] \cdot [\![r_2]\!] = [\![\mathcal{N}_{r_1}]\!] \cdot [\![\mathcal{B}_{r_2}]\!] = L_1 \cdot L_2$$

We will provide a construction for an NBW that accepts the language $L_1 \cdot L_2$ using $\mathcal{N}_{r_1}$ and $\mathcal{B}_{r_2}$.
Let $\mathcal{B}_r = (\Sigma, Q_r, Q_{0_r}, \Delta_r, F_r)$ where:

$$Q_r = Q_{r_1} \cup Q_{r_2}$$
$$Q_{0_r} = Q_{0_{r_1}}$$
$$\Delta_r = \Delta_{r_1} \cup \Delta_{r_2} \cup \{(q, \varepsilon, Q_{0_{r_2}}) \mid q \in F_{r_1}\}$$
$$F_r = F_{r_2}$$

Claim: $[\![\mathcal{B}_r]\!] = L_1 \cdot L_2$ We'll prove this by showing two-directional containment:
$[\![\mathcal{B}_r]\!] \subseteq L_1 \cdot L_2$: Let $w \in [\![\mathcal{B}_r]\!]$. Since w got accepted by $\mathcal{B}_r$, that means that for some run ρ: $inf(\rho) \cap F_r \neq \emptyset$. That means that the run visited infinitely many times in some $q_2 \in F_r = F_{r_2} \subseteq Q_{r_2}$ $(*)$. By the definition of Δ_r - ρ moved to q_2 only by visiting first some $q_1 \in F_{r_1}$. Since q_1 is an accepting state of $\mathcal{N}_{r_1}$ - that means that there exists a prefix of ω - $u \in \Sigma^*$ such that $u \in [\![\mathcal{N}_{r_1}]\!] = L_1$. From $(*)$ we'll get that there exists a suffix of ω - $v \in \Sigma^\omega$ such that $v \in [\![\mathcal{B}_{r_2}]\!] = L_2$. Therefore $w = u \cdot v \in L_1 \cdot$.
$L_1 \cdot L_2 \subseteq [\![\mathcal{B}_r]\!]$: Let $w \in L_1 \cdot L_2$. That means there exists a prefix of ω - $u \in L_1 = [\![\mathcal{N}_{r_1}]\!]$ and a suffix of ω - $v \in L_2 = [\![\mathcal{B}_{r_2}]\!]$ such that $w = u \cdot v$. Since $u \in [\![\mathcal{N}_{r_1}]\!]$, there exists a run $\rho_1 = q_1 q_2 ... q_n$ of $\mathcal{N}_{r_1}$ on u such that $q_n \in F_{r_1}$. Since $v \in [\![\mathcal{B}_{r_2}]\!]$, there exists a run $\rho_2 = q'_1 q'_2 ...$ of $\mathcal{B}_{r_2}$ on v such that $inf(\rho_2) \cap F_r \neq \emptyset$ and therefore there exists a state $q_k \in F_r = F_{r_2}$ that is visited infinitely many times in ρ_2. From these two facts and the construction of $\mathcal{B}_r$ as non-deterministic - there exists a run ρ_3 of $\mathcal{B}_r$ on w that visits q_n and visits q_k infinitely many times - and therefore accepts w. So $w \in [\![\mathcal{B}_r]\!]$.

3. $r = r_1^\omega$:
From the induction assumption we'll get that: since r_1 is a regular expression, there exists an NFW $\mathcal{N}_{r_1} = (\Sigma, Q_{r_1}, Q_{0_{r_1}}, \Delta_{r_1}, F_{r_1})$ such that $[\![\mathcal{N}_{r_1}]\!] = [\![r_1]\!] = L_1$. Applying the semantics function on both sides of the equation yields:

$$[\![r]\!] = [\![r_1^\omega]\!] = [\![r]\!]^\omega = L_1^\omega$$

We will provide a construction for an NBW that accepts the language L_1^ω using $\mathcal{N}_{r_1}$. Let $\mathcal{B}_r = (\Sigma, Q_r, Q_{0_r}, \Delta_r, F_r)$ where:

$$Q_r = Q_{r_1} \cup \{q^*\}$$
$$Q_{0_r} = Q_{0_{r_1}}$$
$$\Delta_r = \Delta_{r_1} \cup \{(q_f, \varepsilon, \{q^*\}), ((q^*, \varepsilon), Q_{0_{r_1}}) \mid q_f \in F_{r_1}\}$$
$$F_r = \{q^*\}$$

Claim: $[\![\mathcal{B}_r]\!] = L_1^\omega$. We'll prove this by showing two-directional containment:
$[\![\mathcal{B}_r]\!] \subseteq L_1^\omega$: Let $w \in [\![\mathcal{B}_r]\!]$. Since w got accepted by $\mathcal{B}_r$, that means that for some run ρ: $inf(\rho) \cap F_{r_1} \neq \emptyset$. That means that the run visited infinitely many times in the only accepting state - q^*. By the construction of $\mathcal{B}_r$ - that means that the run visited infinitely many times in states that are accepting in $\mathcal{N}_{r_1}$. That means that w is a word composed of infinitely many words from L_1 - and therefore $w \in L_1^\omega$.
$L_1^\omega \subseteq [\![\mathcal{B}_r]\!]$: Let $w \in L_1^\omega$. That means that w is composed of infinitely many words from L_1. From the construction of $\mathcal{B}_r$, that means that there exists a run ρ that visited infinitely many times in states that are accepting in $\mathcal{N}_{r_1}$ and then visits the accepting state in $\mathcal{B}_r$ - q^*. Since ρ visits the accepting state q^* infinitely many times - $inf(\rho) \cap F_{r_1} \neq \emptyset$ and so $w \in [\![\mathcal{B}_r]\!]$. ■.

2 Question 2

We will provide a counterexample: Let $\Sigma = \{a, b\}$ and let us consider the following ω-regular language:

$$L = \{w \in \Sigma^\omega \mid \text{the number of } a\text{'s in } w \text{ is either even or infinite}\}$$

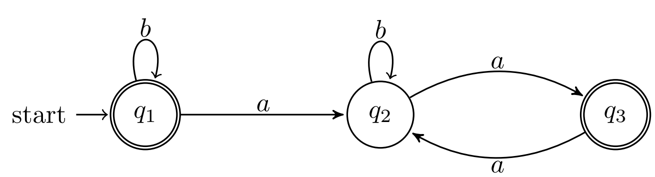
This language is ω-regular as one can see that for $r = (b^*ab^*a)^*b^\omega \cup \Sigma^*(\Sigma^*a\Sigma^*)^\omega$:

$$[\![r]\!] = [\![(b^*ab^*a)^*b^\omega \cup \Sigma^*(\Sigma^*a\Sigma^*)^\omega]\!] = L$$

Let us construct a DBW that accepts L in the following manner: $\mathcal{B} = (\Sigma, Q, q_0, \delta, F)$ where:

$$Q = \{q_1, q_2, q_3\}$$
$$q_0 = q_1$$
$$\delta(q_1, b) = q_1 \ ; \ \delta(q_1, a) = q_2 \ ; \ \delta(q_2, b) = q_2$$
$$\delta(q_2, a) = q_3; \ ; \delta(q_3, b) = q_3 \ ; \ \delta(q_3, a) = q_2$$
$$F = \{q_1, q_3\}$$

Let us draw $\mathcal{B}$:



Now, let $u = \epsilon$ and $v = a$. So $uv = a$ and $|v| = 1$. One can see that $\mathcal{B}$ is a minimal DBW L but any run on $uv^\omega = a^\omega$ induces a sequence of states with a cycle of length $2 > 1 = |v|$.

3 Question 3

The claim is correct. To prove so, we'll show first that $\mathbb{DBGW} = \mathbb{DBW}$. It is trivial that $\mathbb{NBW} \subseteq \mathbb{NBGW}$ (as an $\mathbb{NBW}$ is a specific case of a $\mathbb{NBGW}$ that has one set of accepting states). We showed in class that $\mathbb{DBW} \subsetneq \mathbb{NBW}$, so we'll get: $\mathbb{DBGW} = \mathbb{DBW} \subsetneq \mathbb{NBW} = \mathbb{NBGW}$ that corresponds to $\mathbb{DBGW} \subsetneq \mathbb{NBGW}$.

Lemma: $\mathbb{DBGW} = \mathbb{DBW}$: We'll prove this by showing two-directional containment:

$\mathbb{DBW} \subseteq \mathbb{DBGW}$: This side is trivial as a DBW is a specific case of a DBGW that has one set of accepting states.

$\mathbb{DBGW} \subseteq \mathbb{DBW}$: We will provide a construction that converts a DBGW to a DBW:

Let $\mathcal{G} = (\Sigma, Q, q_0, \delta, \{F_1, ..., F_n\})$ be a DBGW. Let $\mathcal{B} = (\Sigma, Q_b, q_{0_b}, \delta_b, F_b)$ such that:

$$Q_b = Q \times \{1, ..., n\}$$
$$q_{0_b} = (q_0, 1)$$
$$\delta_b = \{((q,i), \sigma, (q', j)) \mid (q, \sigma, q') \in \delta, \text{ if } q \in F_i : j = ((i+1) \ mod \ n) \text{ else } j = i\}$$
$$F_b = F_1 \times \{1\}$$

Claim: $[\![\mathcal{G}]\!] = [\![\mathcal{B}]\!]$ We'll prove this by showing two-directional containment:

$[\![\mathcal{G}]\!] \subseteq [\![\mathcal{B}]\!]$: Let $w \in [\![\mathcal{G}]\!]$. Since w got accepted by $\mathcal{G}$ - by the generalized Büchi automaton acceptance condition that means that there exists a run ρ_G of $\mathcal{G}$ on w such that:

$$\forall i \in \{1, ..., n\} \ ; \ inf(\rho_G) \cap F_i \neq \emptyset$$
$$\rightarrow \forall i \in \{1, ..., n\} \ ; \ \exists q_i \in Q \ : \ q_i \in inf(\rho_G) \cap F_i$$
$$\rightarrow \forall i \in \{1, ..., n\} \ ; \ \exists q_i \in Q \ : \ q_i \in inf(\rho_G) \wedge q_i \in F_i$$

That means that there exist in ρ_G infinitely many configurations of the form:

$$(q_i, \sigma_i u_i) \overset{\sigma_i}{\Rightarrow} (q_{j_i}, u_i)$$

for all $i \in \{1, ..., n\}$ when u_i is a suffix of w and $\sigma_i \in \Sigma$, such that $q_i \in F_i$. By that fact and the construction of $\mathcal{B}$, that means that there exist a run ρ_B with infinitely many configurations of the form:

$$((q_i, i), \sigma_i u_i) \overset{\sigma_i}{\Rightarrow} ((q_{k_i}, ((i+1) \ mod \ n), u_i)$$

for all $i \in \{1, ..., n\}$. That means that specifically, for $i = 1$ there are infinitely many configurations in ρ_B of the form:

$$((q_1, 1), \sigma_1 u_1) \overset{\sigma_1}{\Rightarrow} ((q_{k_1}, (2 \ mod \ n), u_1)$$

such that $q_1 \in F_1$. That means that $(q_1, 1) \in inf(\rho_B)$. Since $q_1 \in F_1$ - we have that $(q_1, 1) \in F_1 \times \{1\} = F_b$ and thus $inf(\rho_B) \cap F_b \neq \emptyset$. Therefore - $w \in [\![\mathcal{B}]\!]$.

$[\![\mathcal{B}]\!] \subseteq [\![\mathcal{G}]\!]$: Let $w \in [\![\mathcal{B}]\!]$. Since w got accepted by $\mathcal{B}$ - that means that there exists a run ρ_B of $\mathcal{B}$ on w such that:

$$inf(\rho_B) \cap F_b = inf(\rho_B) \cap F_1 \times \{1\} \neq \emptyset$$
$$\rightarrow \exists (q_1, 1) \in Q_b \ : \ (q_1, 1) \in inf(\rho_B) \cap F_1 \times \{1\}$$
$$\rightarrow \exists (q_1, 1) \in Q_b \ : \ (q_1, 1) \in inf(\rho_B) \wedge (q_1, 1) \in F_1 \times \{1\}$$

when $q_1 \in F_1$. That means that ρ_B visits infinitely many times in $(q_1, 1)$. By the construction of $\mathcal{B}$ - that means that there are infinitely many configurations in ρ_B of the form:

$$((q_1, 1), \sigma_1 \sigma_2 ... \sigma_n u) \overset{\sigma_1}{\Rightarrow} ((q_{j_1}, ((1+1) \ mod \ n), \sigma_2 ... \sigma_n u) = ((q_{j_1}, (2 \ mod \ n), \sigma_2 ... \sigma_n u) = ((q_{j_1}, 2, \sigma_2 ... \sigma_n u)$$

assuming without loss of generality that $n > 2$, when u is a suffix of w and for all $i \in \{1, ..., n\}$; $\sigma_i \in \Sigma$, such that $q_1 \in F_1$. By the construction of $\mathcal{B}$ - since ρ_B visits infinitely many times in $(q_1, 1)$, there must be a configurations in ρ_B of the form:

$$((q_{j_1}, 2, \sigma_2 ... \sigma_n u) \overset{*}{\Rightarrow} ((q_1, 1), u')$$

where u' is a suffix of u. Once again by the construction of $\mathcal{B}$ - that means that for all $i \in \{1, ..., n\}$ - ρ_B visits infinitely many times in (q_i, i) and by the definition of δ_b we get that there for all $i \in \{1, ..., n\}$ there exist $q_i \in F_i$. Therefore, once again by the definition of δ_b - there exists a run ρ_G with infinitely many configurations of the form:

$$(q_i, \sigma_i u_i) \overset{\sigma_i}{\Longrightarrow} (q_{k_i}, u_i)$$

for all $i \in \{1, ..., n\}$ when u_i is a suffix of w and $\sigma_i \in \Sigma$, such that $q_i \in F_i$. That means that $\forall i \in \{1, ..., n\}$; $inf(\rho_G) \cap F_i \neq \emptyset$ and by the generalized Büchi automaton acceptance condition that means that $w \in [\![\mathcal{B}]\!]$. ∎

4 Question 4

Let $\Sigma_n = \{0, 1, ..., n-1\}$ and let $\oplus_n$ denote addition modulo n. Let:

$$L_n = \left\{ w \in \Sigma_n^\omega \;\middle|\; \exists k \in \Sigma_n \;:\; \text{the letter } k \text{ appears finitely often in } w \right.$$
$$\left. \text{and the letter } k \oplus_n 1 \text{ appears infinitely often in } w \right\}$$

We will provide a an ω-automaton $\mathcal{A}$ such that $[\![\mathcal{A}]\!] = L_n$ that has $O(n)$ states. We will choose the NRW - nondeterministic Rabin automaton $\mathcal{A} = (\Sigma_n, Q, Q_0, \Delta, R)$ where:

$$Q = \Sigma_n = \{0, 1, ..., n-1\}$$
$$Q_0 = Q$$
$$\Delta = \{(i, \sigma, \sigma) \mid i \in Q = \Sigma_n, \sigma \in \Sigma_n\}$$
$$R = \{(\{k \oplus_n 1\}, \{k\}) \mid k \in Q\}$$

One can see that $|Q| = O(n)$.
Correctness argument: The states of the automaton are all the letters in Σ_n - the numbers from 0 to $n-1$. Given a word $w \in \Sigma_n^\omega$, a run ρ of $\mathcal{A}$ on w will pass through the states corresponding to the letters in the word - as defined by the transition function Δ. By the definition of the Rabin acceptance condition, given $R' = \{(G_i, B_i) \mid \forall\, 1 \leq i \leq k \;:\; G_i, B_i \subseteq Q\}$, - a run ρ' is accepting iff:

$$\exists i \;:\; inf(\rho') \cap G_i \neq \emptyset \wedge inf(\rho') \cap B_i = \emptyset$$

By that and the definition of R - ρ is accepting iff:

$$\exists k \;:\; inf(\rho') \cap \{k \oplus_n 1\} \neq \emptyset \wedge inf(\rho') \cap \{k\} = \emptyset$$

That means that ρ is accepting iff it passes infinitely many times in $k \oplus_n 1$ and finitely many times in k, and that corresponds exactly to the condition for w to be in L_n.

5 Question 5

Let R_1 and R_2 be finitary properties, as in $R_1, R_2 \subseteq \Sigma^*$.

5.1 Section a

In this section we are to show that recurrence properties are closed under union. Let us recall that a recurrence property of a finitary set V is an infinitary property W that contains all the infinite words that have infinite prefixes in V, as in $W = \mathcal{R}_{Pref}(V) = \{w \in \Sigma^\omega \mid \forall i \exists j > i \ : \ w[...j] \in V\}$. So let us assume that there exist two recurrence properties P_1 and P_2 such that:

$$P_1 = \mathcal{R}_{Pref}(R_1)$$
$$P_2 = \mathcal{R}_{Pref}(R_2)$$

We are to show that $P_1 \cup P_2$ is also a recurrence property.
Claim: $P_1 \cup P_2 = \mathcal{R}_{Pref}(R_1 \cup R_2)$ We'll prove so by simultaneous two-directional containment. Let $w \in \Sigma^\omega$.

$$w \in P_1 \cup P_2 \Leftrightarrow w \in \mathcal{R}_{Pref}(R_1) \cup \mathcal{R}_{Pref}(R_2) \Leftrightarrow w \in \mathcal{R}_{Pref}(R_1) \vee w \in \mathcal{R}_{Pref}(R_2)$$
$$\Leftrightarrow w \in \{w' \in \Sigma^\omega \mid \forall i \exists j_1 > i \ : \ w'[...j_1] \in R_1\} \vee w \in \{w' \in \Sigma^\omega \mid \forall i \exists j_2 > i \ : \ w'[...j_2] \in R_2\}$$
$$\Leftrightarrow \forall i \exists j_1 \ : \ j_1 > i \ : \ w[...j_1] \in R_1 \vee \forall i \exists j_2 > i \ : \ w[...j_2] \in R_2 \Leftrightarrow \forall i \exists j = \max\{j_1, j_2\} > i \ : \ w[...j] \in R_1 \cup R_2$$
$$\Leftrightarrow w \in \{w' \in \Sigma^\omega \mid \forall i \exists j > i \ : \ w'[...j] \in R_1 \cup R_2\} \Leftrightarrow w \in \mathcal{R}_{Pref}(R_1 \cup R_2) \ \blacksquare$$

5.2 Section b

In this section we are to show that $\mathcal{R}_{Pref}(R_1) \cap \mathcal{R}_{Pref}(R_2) \neq \mathcal{R}_{Pref}(R_1 \cap R_2)$. To do so, we'll provide a counterexample. Let us consider $\Sigma = \{a\}$ and:

$$R_1 = \{a^i \mid i \text{ is prime }\}$$
$$R_2 = \{a^i \mid i \text{ is not prime }\}$$

Of course, $R_1 \cap R_2 = \emptyset$ so by definition $\mathcal{R}_{Pref}(R_1 \cap R_2) = \mathcal{R}_{Pref}(\emptyset) = \emptyset$.
Claim: $a^\omega \in \mathcal{R}_{Pref}(R_1) \cap \mathcal{R}_{Pref}(R_2)$ Since there are infinitely many prime numbers:

$$\forall i \in \mathbb{N} \ : \ \exists j > i \ : \ j \text{ is prime } \rightarrow \forall i \exists j > i \ : \ a^\omega[...j] = a^j \in R_1 \rightarrow a^\omega \in \mathcal{R}_{Pref}(R_1)$$

And since there are infinitely many non-prime numbers:

$$\forall i \in \mathbb{N} \ : \ \exists j > i \ : \ j \text{ is not prime } \rightarrow \forall i \exists j > i \ : \ a^\omega[...j] = a^j \in R_2 \rightarrow a^\omega \in \mathcal{R}_{Pref}(R_2)$$

So $a^\omega \in \mathcal{R}_{Pref}(R_1) \cap \mathcal{R}_{Pref}(R_2)$ $\blacksquare$. Finally, we get $\mathcal{R}_{Pref}(R_1 \cap R_2) \neq \mathcal{R}_{Pref}(R_1) \cap \mathcal{R}_{Pref}(R_2)$.

5.3 Section c

Let:

$$minex(R_1, R_2) = \left\{u_2 \in R_2 \;\middle|\; \exists u_1 \in R_1 \ : \ u_1 \prec u_2 \wedge \nexists u_2' \in R_2 \ : \ u_1 \prec u_2' \prec u_2\right\}$$

Let us observe that $minex(R_1, R_2)$ is the language of all the words from R_2 that have a proper prefix u_1 in R_1 and are minimal in that property, in a sense that for all $u_2 \in minex(R_1, R_2)$ there aren't any other words from R_2 that have the same proper prefix u_1 and are a proper prefix of u_2.
Claim: $\mathcal{R}_{Pref}(R_1) \cap \mathcal{R}_{Pref}(R_2) = \mathcal{R}_{Pref}(minex(R_1, R_2))$ We'll prove so by two-directional containment.
$\mathcal{R}_{Pref}(R_1) \cap \mathcal{R}_{Pref}(R_2) \subseteq \mathcal{R}_{Pref}(minex(R_1, R_2))$: Let $w \in \mathcal{R}_{Pref}(R_1) \cap \mathcal{R}_{Pref}(R_2)$. Then:

$$w \in \mathcal{R}_{Pref}(R_1) \wedge w \in \mathcal{R}_{Pref}(R_2)$$
$$\Longrightarrow w \in \{w' \in \Sigma^\omega \mid \forall i \exists j_1 > i \ : \ w'[...j_1] \in R_1\} \wedge w \in \{w' \in \Sigma^\omega \mid \forall i \exists j_2 > i \ : \ w'[...j_2] \in R_2\}$$
$$\Longrightarrow \forall i \exists j_1 > i \ : \ w[...j_1] \in R_1 \wedge \forall i \exists j_2 > i \ : \ w[...j_2] \in R_2$$

Since for all $i \in \mathbb{N}$, there exists some index $j_1 \in \mathbb{N}$ such that $w[...j_1] \in R_1$ and some index $j_2 \in \mathbb{N}$ such that $w[...j_2] \in R_2$, $minex(R_1, R_2)$ will depend on the relation between j_1 and j_2.
Let $i \in \mathbb{N}$. Let us split into two cases:

1. If the corresponding indices j_1, j_2 hold that there isn't any $j_3 \in \mathbb{N}$ such that: $j_1 < j_3 < j_2$ and $w[...j_1] \prec w[...j_3] \prec w[...j_2]$ then by the definition of *minex* we'll have that $w[...j_2] \in minex(R_1, R_2)$. This holds for all $i \in \mathbb{N}$, so we'll have by the definition of $\mathcal{R}_{Pref}$ that $w \in \mathcal{R}_{Pref}(minex(R_1, R_2))$.
2. If there exists $j_3 \in \mathbb{N}$ such that: $j_1 < j_3 < j_2$ and $w[...j_1] \prec w[...j_3] \prec w[...j_2]$ then let j_3^* be the minimal index that holds for that condition. Then by the definition of *minex* we'll have that $w[...j_3^*] \in minex(R_1, R_2)$. This once again holds for all $i \in \mathbb{N}$, then we'll have by the definition of $\mathcal{R}_{Pref}$ that $w \in \mathcal{R}_{Pref}(minex(R_1, R_2))$.

$\underline{\mathcal{R}_{Pref}(minex(R_1, R_2)) \subseteq \mathcal{R}_{Pref}(R_1) \cap \mathcal{R}_{Pref}(R_2)}$: Let $w \in \mathcal{R}_{Pref}(minex(R_1, R_2))$. Then:

$$\begin{gathered} w \in \mathcal{R}_{Pref}(minex(R_1, R_2)) \\ \Longrightarrow w \in \{w' \in \Sigma^\omega \mid \forall i \exists j > i \ : \ w'[...j] \in minex(R_1, R_2)\} \\ \Longrightarrow \forall i \exists j > i \ : \ w[...j] \in minex(R_1, R_2) \\ \Longrightarrow \forall i \exists j > i \ : \ w[...j] \in R_2 \ : \ \exists u_1 \in R_1 \ : \ u_1 \prec w[...j] \wedge \nexists u_2' \in R_2 \ : \ u_1 \prec u_2' \prec w[...j] \\ \Longrightarrow \forall i \exists j > i \ : \ w[...j] \in R_2 \ : \ \exists k < j \ : \ w[...k] \in R_1 \ : \ w[...k] \prec w[...j] \wedge \nexists u_2' \in R_2 \ : \ w[...k] \prec u_2' \prec w[...j] \end{gathered}$$

Let $i \in \mathbb{N}$. The corresponding indices j, k hold that $j > i$ and $j > k$. Let us split into two cases:

1. If $k \geq i$ then we have:

$$\begin{gathered} w[...j] \in R_2 \wedge w[...k] \in R_1 \ : \ w[...k] \prec w[...j] \wedge \nexists u_2' \in R_2 \ : \ w[...k] \prec u_2' \prec w[...j] \\ \Longrightarrow w \in \{w' \in \Sigma^\omega \mid \forall i \exists j_1 > i \ : \ w'[...j_1] \in R_1\} \wedge w \in \{w' \in \Sigma^\omega \mid \forall i \exists j_2 > i \ : \ w'[...j_2] \in R_2\} \\ \Longrightarrow w \in \mathcal{R}_{Pref}(R_1) \wedge w \in \mathcal{R}_{Pref}(R_2) \\ \Longrightarrow w \in \mathcal{R}_{Pref}(R_1) \cap \mathcal{R}_{Pref}(R_2) \end{gathered}$$

2. If $k < i$ then let us observe that there exists $j' > j$ such that:

$$\begin{gathered} \exists w[...j'] \in minex(R_1, R_2) \\ \Longrightarrow \exists m < j' \ : \ w[...m] \in R_1 \ : \ w[...m] \prec w[...j'] \wedge \nexists u_2' \in R_2 \ : \ w[...m] \prec u_2' \prec w[...j'] \end{gathered}$$

Let us assume towards contradiction that $m < i$. Then we'll get that:

$$\exists w[...i] \in R_2 \ : \ w[...m] \prec w[...i] \prec w[...j']$$

thus contradicting the former reasoning. Therefore we have that there exists $m \in \mathbb{N}$ such that $m \geq i$ and:

$$\begin{gathered} w[...j] \in R_2 \wedge w[...m] \in R_1 \ : \ w[...m] \prec w[...j] \wedge \nexists u_2' \in R_2 \ : \ w[...m] \prec u_2' \prec w[...j] \\ \Longrightarrow w \in \{w' \in \Sigma^\omega \mid \forall i \exists j_1 > i \ : \ w'[...j_1] \in R_1\} \wedge w \in \{w' \in \Sigma^\omega \mid \forall i \exists j_2 > i \ : \ w'[...j_2] \in R_2\} \\ \Longrightarrow w \in \mathcal{R}_{Pref}(R_1) \wedge w \in \mathcal{R}_{Pref}(R_2) \\ \Longrightarrow w \in \mathcal{R}_{Pref}(R_1) \cap \mathcal{R}_{Pref}(R_2) \ \blacksquare \end{gathered}$$

5.4 Section d

In this section we are to show that recurrence properties are closed under intersection. In the last section we proved that given two finitary properties R_1 and R_2 - the intersection of their corresponding recurrence properties $P_1 = \mathcal{R}_{Pref}(R_1)$ and $P_2 = \mathcal{R}_{Pref}(R_2)$, as in $\mathcal{R}_{Pref}(R_1) \cap \mathcal{R}_{Pref}(R_2)$ is a recurrence relation of the finitary property $minex(R_1, R_2)$, and so using this construction - the recurrence property is closed under intersection. $\blacksquare$

5.5 Section e

In this section we are to show that persistence properties are closed under union and intersection. Let R_1 and R_2 be finitary properties, as in $R_1, R_2 \subseteq \Sigma^*$. From the duality properties of the linguistic characterizations we saw in class, we know that for a finitary property R:

$$(*)\ \overline{\mathcal{R}_{Pref}(R)} = \mathcal{P}_{Pref}(\overline{R})$$

<u>Closure under union:</u> Let us consider the $\mathcal{P}_{Pref}(R_1) \cup \mathcal{P}_{Pref}(R_2)$. From $(*)$:

$$\mathcal{P}_{Pref}(R_1) \cup \mathcal{P}_{Pref}(R_2) = \overline{\mathcal{R}_{Pref}(\overline{R_1})} \cup \overline{\mathcal{R}_{Pref}(\overline{R_2})}$$

From De Morgan's laws:

$$\overline{\mathcal{R}_{Pref}(\overline{R_1})} \cup \overline{\mathcal{R}_{Pref}(\overline{R_2})} = \overline{\mathcal{R}_{Pref}(\overline{R_1}) \cap \mathcal{R}_{Pref}(\overline{R_2})}$$

From the construction shown in the previous sections for an intersection of two recurrence properties:

$$\mathcal{R}_{Pref}(\overline{R_1}) \cap \mathcal{R}_{Pref}(\overline{R_2}) = \mathcal{R}_{Pref}(minex(\overline{R_1}, \overline{R_2}))$$
$$\Longrightarrow \overline{\mathcal{R}_{Pref}(\overline{R_1}) \cap \mathcal{R}_{Pref}(\overline{R_2})} = \overline{\mathcal{R}_{Pref}(minex(\overline{R_1}, \overline{R_2}))}$$

From $(*)$ again:

$$\overline{\mathcal{R}_{Pref}(minex(\overline{R_1}, \overline{R_2}))} = \mathcal{P}_{Pref}(\overline{minex(\overline{R_1}, \overline{R_2})})$$

So finally:

$$\mathcal{P}_{Pref}(R_1) \cup \mathcal{P}_{Pref}(R_2) = \mathcal{P}_{Pref}(\overline{minex(\overline{R_1}, \overline{R_2})})$$

So we saw a construction for a closure to a union of two persistence properties. ■

<u>Closure under intersection:</u> Let us consider the $\mathcal{P}_{Pref}(R_1) \cap \mathcal{P}_{Pref}(R_2)$. From $(*)$:

$$\mathcal{P}_{Pref}(R_1) \cap \mathcal{P}_{Pref}(R_2) = \overline{\mathcal{R}_{Pref}(\overline{R_1})} \cap \overline{\mathcal{R}_{Pref}(\overline{R_2})}$$

From De Morgan's laws:

$$\overline{\mathcal{R}_{Pref}(\overline{R_1})} \cap \overline{\mathcal{R}_{Pref}(\overline{R_2})} = \overline{\mathcal{R}_{Pref}(\overline{R_1}) \cup \mathcal{R}_{Pref}(\overline{R_2})}$$

From the construction shown in the previous sections for a union of two recurrence properties:

$$\mathcal{R}_{Pref}(\overline{R_1}) \cup \mathcal{R}_{Pref}(\overline{R_2}) = \mathcal{R}_{Pref}(\overline{R_1} \cup \overline{R_2})$$
$$\Longrightarrow \overline{\mathcal{R}_{Pref}(\overline{R_1}) \cup \mathcal{R}_{Pref}(\overline{R_2})} = \overline{\mathcal{R}_{Pref}(\overline{R_1} \cup \overline{R_2})}$$

From $(*)$ and De Morgan's laws:

$$\overline{\mathcal{R}_{Pref}(\overline{R_1} \cup \overline{R_2})} = \mathcal{P}_{Pref}(\overline{\overline{R_1} \cup \overline{R_2}}) = \mathcal{P}_{Pref}(R_1 \cap R_2)$$

So finally we have:

$$\mathcal{P}_{Pref}(R_1) \cap \mathcal{P}_{Pref}(R_2) = \mathcal{P}_{Pref}(R_1 \cap R_2)$$

So we saw a construction for a closure to an intersection of two persistence properties. ■

6 Question 6

Let $L \subseteq \Sigma^\omega$ be an infinitary language. Let us consider the following definition of a finite words relation: For $x, y \in \Sigma^*$ we have that:

$$x \equiv_L y \Leftrightarrow \forall z \in \Sigma^\omega \ :\ xz \in L \Leftrightarrow yz \in L$$

6.1 Section i

In this section we are to prove that the relation $\equiv_L$ is an equivalence relation. To do so, by definition, we'll need to show that $\equiv_L$ is transitive, reflexive and symmetric.
Transitivity: Let $x, y, z \in \Sigma^*$ and let us assume that $x \equiv_L y$ and $y \equiv_L z$. We have to show that $x \equiv_L z$. Since $x \equiv_L y$, by definition we have that:

$$(*)\ \forall \psi \in \Sigma^\omega\ :\ x\psi \in L \Leftrightarrow y\psi \in L$$

Since $y \equiv_L z$, by definition we have that:

$$(**)\ \forall \psi \in \Sigma^\omega\ :\ y\psi \in L \Leftrightarrow z\psi \in L$$

Let $\psi \in \Sigma^\omega$ and let us assume that $x\psi \in L$. From $(*)$ we'll get $y\psi \in L$. From $(**)$ we'll get $z\psi \in L$. Now let us assume that $x\psi \notin L$. From $(*)$ we'll get $y\psi \notin L$. From $(**)$ we'll get $z\psi \notin L$. So we got $x\psi \in L \Leftrightarrow z\psi \in L$ and by definition $x \equiv_L z$.
Reflexivity: Let $x \in \Sigma^*$. We have to show that $x \equiv_L x$. It is obvious that:

$$\forall \psi \in \Sigma^\omega\ :\ x\psi \in L \Leftrightarrow x\psi \in L$$

so $x \equiv_L x$.
Syymertricity: Let $x, y \in \Sigma^*$ and let us assume that $x \equiv_L y$. We have to show that $y \equiv_L z$. Since $x \equiv_L y$, by definition we have that:

$$\forall \psi \in \Sigma^\omega\ :\ x\psi \in L \Leftrightarrow y\psi \in L$$

That of course means that:

$$\forall \psi \in \Sigma^\omega\ :\ y\psi \in L \Leftrightarrow x\psi \in L$$

so we have that $y \equiv_L x$. ■

6.2 Section ii

In this section we are to prove or refute the following claim: If L is accepted by a DBA then the number of equivalence classes in $\equiv_L$ is finite.
Claim: The claim is correct To prove so, let $L \subseteq \Sigma^\omega$ and let us assume that L is accepted by a DBW $\mathcal{D} = (\Sigma, Q, q_0, \delta, F)$, as in $[\![\mathcal{D}]\!] = L$. Let us denote for any finite word $w \in \Sigma^*$: r_w to be the only run of $\mathcal{D}$ on w (due to $\mathcal{D}$ being deterministic) and q_w to be the final state in that run (which exists because w is final and $\mathcal{D}$ is deterministic), as in $r_w = q_0 q_1 ... q_w$. Moreover, let us denote for any infinite word $w \in \Sigma^\omega$: ρ_w to be the only run of $\mathcal{D}$ on w and let us call a *sub-run* a partial run of some run.
Lemma: For all $x, y \in \Sigma^\omega$ if $x \not\equiv_L y$ then $q_x \neq q_y$.
To prove so, we'll assume towards contradiction that $q_x = q_y$. Since $x \not\equiv_L y$ that means that (without loss of generality) there exists $z \in \Sigma^*$ such that $xz \in L$ and $yz \notin L$. Since we assumed that $[\![\mathcal{D}]\!] = L$, that means that xz is accepted by $\mathcal{D}$ while yz is not. Since we assumed that $q_x = q_y$, that means that the runs of $\mathcal{D}$ on x and y - r_x and r_y respectively are:

$$(q_0, xz) \overset{*}{\Rightarrow} (q_x, z)$$
$$(q_0, yz) \overset{*}{\Rightarrow} (q_y, z)$$

and since $q_x = q_y$:

$$(q_0, xz) \overset{*}{\Rightarrow} (q_x, z)$$
$$(q_0, yz) \overset{*}{\Rightarrow} (q_x, z)$$

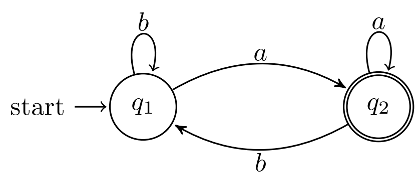
Since $xz \in L$, then it holds that the $inf(\rho_{xz}) \cap F \neq \emptyset$. That means that there exists an accepting state $q_f \in F$ such that the run ρ_{xz} visits it infinitely many times. Since r_x is final, that means that the sub-run of ρ_{xz} after r_x also visits infinitely many times in q_f. Since $q_x = q_y$, the sub-run of ρ_{yz} after r_y is that same as the sub-run of ρ_{xz} after r_x - so the sub-run of ρ_{yz} after r_y also visits infinitely many times in q_f and since r_y is final, that means that ρ_{yz} also visits infinitely many times in q_f and so we get that $yz \in [\![\mathcal{D}]\!] = L$, contradicting that $yz \notin L$ ■.
Let us now return to the original proof: let us assume towards contradiction that the number of equivalence classes in $\equiv_L$ is infinite. That means that there exist infinitely many words $w_1, w_2, ... \in \Sigma^*$ such that $\forall i \neq j \ : \ w_i \not\equiv w_j$. From the lemma we get that since $\forall i \neq j \ : \ w_i \not\equiv w_j$ - $\forall i \neq j \ : \ q_i \not\equiv q_j$. That means that we get infinitely many different states, contradicting that the number of states in $\mathcal{D}$ is final. ■

6.3 Section iii

In this section we are to prove or refute the following claim: If L is accepted by a DBA then the number of states in a minimal DBA is equivalent to the number of equivalence classes in $\equiv_L$.
Claim: The claim is incorrect To prove so, we'll provide a counterexample. Let $\mathcal{D} = (\Sigma, Q, q_0, \delta, F)$ be a DBA such that:

$$Q = \{q_1, q_2\}$$
$$q_0 = q_1$$
$$\delta(q_1, b) = q_1 \ ; \ \delta(q_1, a) = q_2$$
$$\delta(q_2, b) = q_1 \ ; \ \delta(q_2, a) = q_2$$
$$F = \{q_2\}$$

Let us draw $\mathcal{D}$:



Let $L = \Sigma^* a^\omega$. One can see that $[\![\mathcal{D}]\!] = L$, as in the language $\mathcal{D}$ accepts is the language of all words that have infinite a's in them. Since the condition of infinite a' must be checked with an accepting state, there has to be an additional state for words that does not have infinite a' in them. So the minimal number of states to accept L is 2 - the same number as in $\mathcal{D}$ and so it is a minimal DBA for L. Since $\mathcal{D}$ only accepts words with infinite a's, it has only one equivalence class, which is less that the number of states in a minimal DBA that accepts it. So the claim is incorrect. ■