Intro to AI Theoretical HW 2

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1 Planning - POP (Partial Order Planning) Algorithm

a) POP Algorithm

Let us go over the steps of the POP algorithm:

1. **Make-Minimal-Plan:**
   Let us define the initial state of the plan:
   - Preconditions: None
   - Effects: $On(A, B)$, $On(B, Table), On(C, Table)$, $Clear(A)$, $Clear(C)$, $Color(A, Red)$, $Color(B, Green)$, $Color(C, Blue)$, $PaintAvilibale$

   And the goal state:
   - Preconditions: $On(x, y)$, $Color(x, Blue)$, $Color(y, Blue)$
   - Effects: None

   Let us also define a constraint that the initial state comes before the goal state.

2. **Main-Loop:**

   (a)
   - **If-Solution?**:
     Since not all preconditions are fulfilled - the plan is not complete.
   - **Select-Subgoal**:
     Let us choose an unsatisfied precondition of the goal state to state the planning process with, say $Color(x, Blue)$.
   - **Choose-Operator**:
     Note that the initial state has an effect $Color(C, Blue)$. Thus with the assignment $x = C$, a causal link can be added between these two conditions, with the partial ordering already set between the start and goal states.
   - **Resolve-Threats**:
     By the addition of the link, no clobbers were added.

   (b)
   - **If-Solution?**:
     Since not all preconditions are fulfilled - the plan is not complete.
   - **Select-Subgoal**:
     Let us now choose, say $Color(y, Blue)$.
   - **Choose-Operator**:
     Since the initial state has no other effect that can fit here, we now have to add an additional step to the plan that results with the a colored block. The action $Paint(Block, Blue)$ is fitting here with the assignment $Block = y$. The preconditions for this action are $Clear(y)$

and $PaintAvailable$. With the assignment $y = A$, we'll have both preconditions from the effects of the initial state. So in conclusion, the added step is thus termed $Paint(A, Blue)$ with:

- Preconditions: $Clear(A)$, $PaintAvailable$
- Effects: $Color(A, Blue)$, *not* $PaintAvilibale$

added with the causal link of being before the goal state.

- **Resolve-Threats**:
  By the addition of the step and link, no clobbers were added.

(c)

- **If-Solution?**:
  Since not all preconditions are fulfilled - the plan is not complete.
- **Select-Subgoal**:
  Let us now choose the precondition $Clear(A)$ of $Paint(A, Blue)$.
- **Choose-Operator**:
  The initial state has an effect $Clear(A)$, so let us add a causal link between them.
- **Resolve-Threats**:
  By the addition of the link, no clobbers were added.

(d)

- **If-Solution?**:
  Since not all preconditions are fulfilled - the plan is not complete.
- **Select-Subgoal**:
  Let us now choose the precondition $PaintAvilibale$ of $Paint(A, Blue)$.
- **Choose-Operator**:
  The initial state has an effect $PaintAvilibale$, so let us add a causal link between them.
- **Resolve-Threats**:
  By the addition of the link, no clobbers were added.

(e)

- **If-Solution?**:
  Since not all preconditions are fulfilled - the plan is not complete.
- **Select-Subgoal**:
  Let us now choose the second precondition $On(x, y)$ of the goal state, where we already assigned $x = C, y = A$.
- **Choose-Operator**:
  To result in the required precondition, we must add another step, and the fitting action in this case will be $Put(C, A)$. Thus let us add the step $Put(C, A)$ to the plan with:
  - Preconditions: $Clear(C)$, $Clear(A)$, $On(C, z)$.
  - Effects: $On(C, A)$, *not* $Clear(A)$, *not* $On(C, z)$, $Clear(z)$

  added with the causal link between the effect $On(C, A)$ and the precondition $On(x, y)$ of the goal state with the assignments $x = A, y = C$.
- **Resolve-Threats**:
  We now have that the precondition $Clear(A)$ of the action $Paint(A, Blue)$ is with contradiction to the effect *not* $Clear(A)$ of $Put(C, A)$. Thus we will order $Paint(A, Blue)$ before $Put(A, C)$.

(f) , (g), (h) are steps where the preconditions of $Put(C, A)$ - $Clear(C)$, $Clear(A)$, $On(C, z)$ are linked to the corresponding effects of the initial state $Clear(C)$, $Clear(A)$, $On(C, Table)$ accordingly with the assignment $z = Table$.

Since all preconditions are now fulfilled, this concludes the algorithm.

b) Conditional POP Algorithm

We have that the above plan does not take the color of B into consideration so it'll work fine for this case also. That being said, if there's a upside to moving instead of painting, then a better plan can be achieved in this case. Let us then construct a conditional plan using the conditional POP algorithm:

1. **Make-Minimal-Plan:**
   Let us define the initial state of the plan:
   - Preconditions: None
   - Effects: $On(A, B),\ On(B, Table), On(C, Table),\ Clear(A),\ Clear(C),\ Color(A, Red),\ Color(C, Blue),\ PaintAvilibale$

   And the goal state:
   - Preconditions: $On(x, y),\ Color(x, Blue),\ Color(y, Blue)$
   - Effects: None

   Let us also define a constraint that the initial state comes before the goal state.

2. **Main-Loop:**
   First (and the interesting) step:

   (a)
   - **If-Solution?**:
     Since not all preconditions are fulfilled - the plan is not complete.
   - **Select-Subgoal**:
     Let us choose an unsatisfied precondition of the goal state to state the planning process with, say $Color(x, Blue)$.
   - **Choose-Operator**:
     This can be achieved possibly by the positive effect of the operator $Check(Color(z, Blue))$ with the assignment $z = B$. Thus let us add a step for the operator $Check(Color(B, Blue))$:
     - Preconditions: None
     - Possible effects: $Color(B, Blue)$ or *not* $Color(B, Blue)$.

     with an addition of a causal link from the positive case to the precondition $Color(x, Blue)$ of the goal state with the assignment $x = B$. The negative case will be linked to the plan constructed in the last part - where the color of B is not relevant.
   - **Resolve-Threats**:
     By the addition of the step and conditional links, no clobbers were added.

   From here on, using similar steps as shown in the last part, we construct a step for the action $PutOnTable(A)$ and a step for the action $Put(C, B)$ such that at the end C will result on top of B while both being blue.

2 Bayes Networks I

Definition 2.1. *Let X, Y and Z be disjoint sets of random variables. X and Y are **conditionally independent** given Z iff for all $x \in X, y \in Y$ x we have $I(x, y|Z)$ - meaning x is conditionally independent of y given the variables in Z. In this case we write $I(X, Y|Z)$.*

Definition 2.2. *Let $X = \{X_1, ..., X_n\}$ be a set of random variables. A **Bayes Network** over X is a pair $BN = (G, \Theta)$ where:*

- *$G = (X, E)$ is a directed acyclic graph with:*

$$E = \{\langle X_i, X_j \rangle \; |X_i, X_j \in X, i \neq j, \text{ not } I(\{X_i\}, \{X_j\} \mid \{\})\}$$

  meaning the edges in G are between dependant variables.

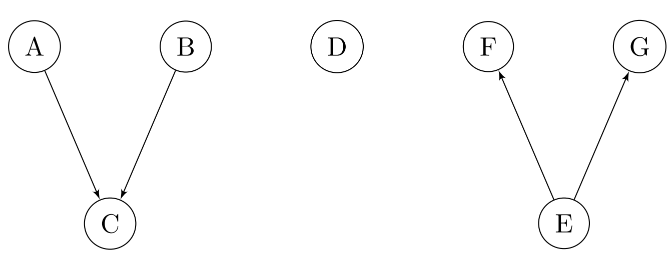
- *Θ is a table that holds the conditional probabilities of all the random variables given their 'parents' in the graph, as in: $Pr(X_j \mid X_i)$ for all $\langle X_i, X_j \rangle \in E$.*

a) Bayes Networks Construction

As we know Bayes Networks includes edges that correspond to conditional probability dependencies, let us construct a Bayes Network consistent with the stated independence statements:

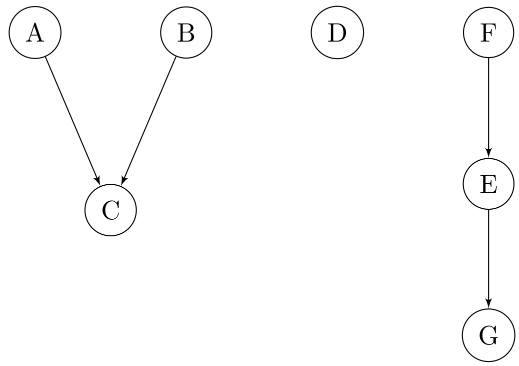
1. The statement $I(\{A, B, C\}, \{D, E, F, G\}|\{\})$ says that the sets of random variables $\{A, B, C\}$ and $\{D, E, F, G\}$ are independant and thus there are no edges between them whatsoever.
2. The statement $I(\{D\}, \{E, F, G\}|\{\})$ says that there are no edges between the singleton $\{D\}$ and the set of vertices $\{E, F, G\}$. Since the previous statement said the D is not connected to the rest of the vertices either, we have that D must be without any edges.
3. The statement $I(\{A\}, \{B\}|\{\})$ says that there are no edges between A and B.
4. The statement $I(\{F\}, \{G\}|\{E\})$ says that F and G are conditionally independant on E - meaning there isn't a direct connection between F and G, but that E is somehow connected to both.
5. The statement *not* $I(\{A\}, \{B\}|\{C\})$ says that A and B are not conditionally independant on C, or in other terms - are conditionally dependant on C. Since by the third statement we know there are no edges between A and B, we must have that there is a 'converging' connection where A and B converge to C.
6. The statement *not* $I(\{F\}, \{G\}|\{\})$ says that F and G are dependant. Thus, with the fourth statement we have that either there's a 'pass-through' connection between F, G and E where E is in the middle, or that E 'diverges' to F and G.

One such graph can be the following:

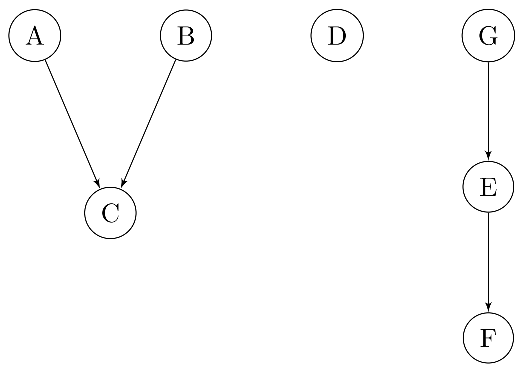


2.1 Uniqueness

As mentioned, we can have a 'pass-through' connection between F, G and E when E is in the middle, so there are two additional possible graphs, for a total of 3. The second one is:

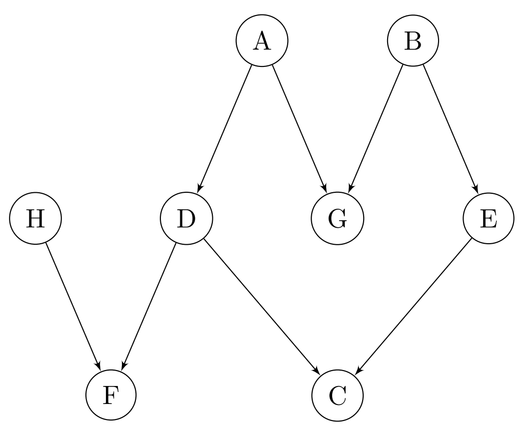


And the third one:



3 Bayes Networks II

Let us draw the described network which we will note as N:



a) Poly-Tree

Definition 3.1. *Let $G = (V, E)$ be a directed graph. Then let $G_U = (V, E_U)$ be the* ***underlying undirected graph*** *of G where E_U is a set that contains all the edges of E but without their direction.*

Definition 3.2. *A directed graph $G = (V, E)$ is a **poly-tree** iff it is acyclic and its underlying undirected graph $G_U = (V, E_U)$ is also acyclic.*

By the above definition, N is not a poly-tree since in the underlying undirected graph there's a cycle A, G, B, E, C, D, A.

b) Singly-Connected

Definition 3.3. *A directed graph $G = (V, E)$ is a **directed path singly-connected graph** iff there is at most one directed path for every pair of vertices in G.*

By the above definition, N is indeed a directed path singly-connected graph since the only undirected cycle in it (A, G, B, E, C, D, A) converges on two different nodes in the directed graph - G and C and thus there are no two vertices with two directed paths between them.

c) d-Separation

Let $BN = (G, \Theta)$ be a Bayes Network and let $G_U = (V, E_U)$ be the underlying undirected graph of $G = (V, E)$. Let $Z \subseteq V$ be a set of vertices.

Definition 3.4. *Let p be a path in G_U. A vertex $w \in p$ is a **blocked vertex** by Z if one of the following holds:*

- *$w \in Z$ and w is a 'pass-through' or a 'diverging' vertex in p with respect to the directed graph G*
- *$w \notin Z$ and neither does any of his descendants in G, and w is a 'converging' vertex in p with respect to the directed graph G*

Definition 3.5. *A path $p \in G_U$ between two vertices $x, y \in V$ is a **blocked path** by Z if there exists $w \in p$ which is a blocked vertex by Z, such that $w \neq x, y$.*

Let $X, Y \subseteq V$ be disjoint sets of vertices that are also disjoint with Z.

Definition 3.6. *A vertex $x \in X$ is a **d-separated vertex** from a vertex $y \in Y$ given Z iff for every simple path $p \in G_U$ between x and y - p is a blocked path by Z. In this case we write $d - sep(x, y \mid Z)$.*

Definition 3.7. *X is **d-separated set** from Y by Z iff for every pair of vertices $x \in X, y \in Y$ we have $d - sep(x, y \mid Z)$. In this case we write $d - sep(X, Y \mid Z)$.*

Theorem 3.8. *If $d - sep(X, Y \mid Z)$ then $I(X, Y \mid Z)$.*

By the above theorem, let us determine the following statements for N:

1. $I(\{D\}, \{E\} | \{\})$
   In this case $Z = \emptyset$. The statement is correct. There are two simple paths between D and E; the path D, C, E is blocked due to C as it is a barren node and thus diverging and of course $C \notin Z$. The path D, A, G, B, E is also blocked due to G which is also barren and not in Z.

2. $I(\{D\}, \{E\} | \{A\})$
   The statement is correct due to the same arguments as above, as $G \notin Z$ also in this case.

3. $I(\{D\}, \{E\} | \{A, C\})$
   The statement is not correct. In this case $C \in Z$ so the path D, C, E is now not blocked.

4. $I(\{B\}, \{F\} | \{G\})$
   The statement is not correct. The path F, D, A, G, B is not blocked, as A is diverging and not in Z, D is pass-through and also not in Z and G is converging but in Z.

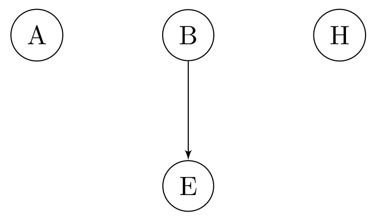
5. $I(\{B\}, \{F\} | \{D\})$
   The statement is correct. There are two simple paths between B and F; the path F, D, A, G, B is blocked by the pass-through vertex D which is in Z and the path F, D, C, E, B is blocked by the converging vertex C which is not in Z.

6. $I(\{B\},\{F\}|\{A,C\})$
   The statement is not correct. The path F,D,C,E,B is not blocked, as D is pass-through and not in Z, C is converging but in Z and E is pass-through but not in Z.

d) Bayesian Inference

As we saw in class, preprocessing in a Bayesian Network gets rid of all barren nodes that are not in the query or evidence and all root nodes without parents that are in the evidence.

We are to calculate $P(B = true, A = true \mid E = true, H = false)$. Let us denote the query variables $Q = \{A, B\}$ and the evidence variables $K = \{E, H\}$.
Now let us remove barren nodes C, G, F - as they are not in Q or K. Doing so causes D to also be barren and so we remove it as well, as it is not in Q or K. These operations yields the following network which we will denote as N':



We see that $d - sep(Q, \{H\} \mid \{\})$, $d - sep(\{A\}, \{B\} \mid K)$ and $d - sep(\{A\}, K \mid \{\})$. Thus:

$$P(B = true, A = true \mid E = true, H = false) =$$
$$P(B = true \mid E = true, H = false) \cdot P(A = true \mid E = true, H = false) =$$
$$P(B = true \mid E = true) \cdot P(A = true) =$$
$$P(E = true \mid B = true) \cdot \frac{P(B = true)}{P(E = true)} \cdot P(A = true) =$$
$$P(E = true \mid B = true) \cdot \frac{P(B = true)}{\sum_{\psi \in \{true, false\}} P(E = true \mid B = \psi) \cdot P(B = \psi)} \cdot P(A = true) =$$
$$0.8 \cdot \frac{0.2}{0.8 \cdot 0.2 + 0.1 \cdot 0.8} \cdot 0.3 = 0.2$$

4 Rational Decisions

For any given policy $\psi \in \{A, B, C, D\}$ let us define it as a triplet $\psi = (\psi_1, \psi_2, \psi_3)$ where ψ_i corresponds to the action of policy ψ at month $1 \le i \le 3$. Let F_{ψ_i} be a random variable that corresponds to the rate of increase factor at the beginning of month i knowing action ψ_i was done up to the beginning of that month. In our case:

$$P(F_{\psi_i} = 5 \mid \psi_i = NO-OP) = 0.9$$
$$P(F_{\psi_i} = 2 \mid \psi_i = NO-OP) = 0.1$$

Let us denote the total number of dead people across 3 months under that policy as a random variable D_ψ. Let us denote $D_\psi = \sum_{i=1}^{3} D_{\psi_i}$ where D_{ψ_i} is a random variable that corresponds to the total number of dead people under policy ψ at the end of month i - meaning after the action ψ_i was performed. In our case:

$$D_{\psi_i} = 0.01 \cdot 30 \cdot \sqrt{R_{\psi_{i-1}} R_{\psi_i}} = 0.3 \cdot \sqrt{R_{\psi_{i-1}} R_{\psi_i}}$$

where R_{ψ_i} is a random variable that corresponds to the daily rate at the end of month i under policy ψ with the initial condition:

$$R_{\psi_0} = 4000$$

for all ψ. By these definitions and by the question description, we have:

$$R_{\psi_i} = \quad F_{\psi_i} \cdot R_{\psi_{i-1}}$$

Thus :

$$D_{\psi_i} = 0.3 \cdot R_{\psi_{i-1}} \sqrt{F_{\psi_i}}$$

And so:

$$D_\psi = 0.3 \sum_{i=1}^{3} R_{\psi_{i-1}} \sqrt{F_{\psi_i}}$$

a) Dominating Policy

For any policy $\psi \neq B$:

$$\begin{aligned}
E[D_\psi] = \quad & E\left[\sum_{i=1}^{3} D_{\psi_i}\right] = \sum_{i=1}^{3} E[D_{\psi_i}] = 0.3 \sum_{i=1}^{3} E\left[R_{\psi_{i-1}} \sqrt{F_{\psi_i}}\right] = \\
& 0.3\left[E\left[R_{\psi_0}\sqrt{F_{\psi_1}}\right] + E\left[R_{\psi_1}\sqrt{F_{\psi_2}}\right] + E\left[R_{\psi_2}\sqrt{F_{\psi_3}}\right]\right] = \\
& 0.3\left[R_{\psi_0} E\left[\sqrt{F_{\psi_1}}\right] + R_{\psi_0} E\left[F_{\psi_1} \cdot \sqrt{F_{\psi_2}}\right] + R_{\psi_0} E\left[F_{\psi_2} \cdot F_{\psi_1} \cdot \sqrt{F_{\psi_3}}\right]\right] = \\
& 0.3 R_{\psi_0}\left[E\left[\sqrt{F_{\psi_1}}\right] + E\left[F_{\psi_1}\right] E\left[\sqrt{F_{\psi_2}}\right] + E\left[F_{\psi_2}\right] E\left[F_{\psi_1}\right] E\left[\sqrt{F_{\psi_3}}\right]\right] = \\
& 1200\left[E\left[\sqrt{F_{\psi_1}}\right] + E\left[F_{\psi_1}\right]\left(E\left[\sqrt{F_{\psi_2}}\right] + E\left[F_{\psi_2}\right] E\left[\sqrt{F_{\psi_3}}\right]\right)\right]
\end{aligned}$$

Thus the expected value depend only on the rates of each policy. But for B:

$$E[D_B] = \quad 1200\left[E\left[\sqrt{F_{B_1}}\right] + E\left[F_{B_1} \cdot \sqrt{F_{B_2}}\right] + E\left[F_{B_1} \cdot F_{B_2} \cdot \sqrt{F_{B_3}}\right]\right]$$

- Policy A:
  This policy results in no loss of BMEMU since nothing is being done and BMEMU is lost only on lockdowns. So BMEMU = 0. Let us then calculate the expected values as described above. Since every month we have the same action which is NO-OP, we have:

$$E\left[\sqrt{F_{A_j}}\right] = \sum_{f_{A_j} \in \{2,5\}} \sqrt{f_{A_j}} \cdot P(F_{A_j} = f_{A_j}) = \sqrt{2} \cdot 0.1 + \sqrt{5} \cdot 0.9 = 2.153$$

$$E\left[F_{A_j}\right] = \sum_{f_{A_j} \in \{2,5\}} f_{A_j} \cdot P(F_{A_j} = f_{A_j}) = 2 \cdot 0.1 + 5 \cdot 0.9 = 4.7$$

  for all $1 \leq j \leq 3$. And thus:

$$E[D_A] = 1200\left[E\left[\sqrt{F_{A_1}}\right] + E\left[F_{A_1}\right]\left(E\left[\sqrt{F_{A_2}}\right] + E\left[F_{A_2}\right]E\left[\sqrt{F_{A_3}}\right]\right)\right] =$$
$$1200[2.153 + 4.7(2.153 + 4.7 \cdot 2.153)] = 71827.674 \approx 71828$$

- Policy B:
  In this case there will be a cost of 100 BMEMU with a probability of 0.9 so the expected value of the BMEMU is 90. Since the first month is NO-OP the first term is as calculated before:

$$E\left[F_{B_1}\right] = 4.7$$

  About the second and third terms:

$$E\left[F_{B_1} \cdot \sqrt{F_{B_2}}\right] = \sum_{f_{B_1}, f_{B_2}} f_{B_1}\sqrt{f_{B_2}} \cdot P(F_{B_1} = f_{B_1}, F_{B_2} = f_{B_2}) =$$
$$\sum_{f_{B_1} \in \{2,5\}} \sum_{f_{B_2}} f_{B_1}\sqrt{f_{B_2}} \cdot P(F_{B_2} = f_{B_2} \mid F_{B_1} = f_{B_1}) \cdot P(F_{B_1} = f_{B_1}) =$$
$$\sum_{f_{B_1} \in \{2,5\}} f_{B_1} P(F_{B_1} = f_{B_1}) \sum_{f_{B_2}} \sqrt{f_{B_2}} \cdot P(F_{B_2} = f_{B_2} \mid F_{B_1} = f_{B_1}) =$$
$$2 \cdot 0.1(\sqrt{2} \cdot 0.1 + \sqrt{5} \cdot 0.9) + 5 \cdot 0.9\sqrt{0.1} \cdot 1 = 1.853$$

$$E\left[F_{B_1} \cdot F_{B_2} \cdot \sqrt{F_{B_3}}\right] = \sum_{f_{B_1}, f_{B_2}, f_{B_3}} f_{B_1} f_{B_2}\sqrt{f_{B_3}} \cdot P(F_{B_1} = f_{B_1}, F_{B_2} = f_{B_2}) =$$
$$\sum_{f_{B_1} \in \{2,5\}} \sum_{f_{B_2}} \sum_{f_{B_3}} f_{B_1} f_{B_2}\sqrt{f_{B_3}} \cdot P(F_{B_3} = f_{B_3} \mid F_{B_2} = f_{B_2}, F_{B_1} = f_{B_1}) \cdot$$
$$P(F_{B_2} = f_{B_2} \mid F_{B_1} = f_{B_1}) \cdot P(F_{B_1} = f_{B_1}) =$$
$$\sum_{f_{B_1} \in \{2,5\}} f_{B_1} P(F_{B_1} = f_{B_1}) \sum_{f_{B_2}} f_{B_2} P(F_{B_2} = f_{B_2} \mid F_{B_1} = f_{B_1})$$
$$\sum_{f_{B_3}} \sqrt{f_{B_3}} \cdot P(F_{B_3} = f_{B_3} \mid F_{B_2} = f_{B_2}, F_{B_1} = f_{B_1}) =$$
$$2 \cdot 0.1 \cdot (2 \cdot 0.1(\sqrt{2} \cdot 0.1 + \sqrt{5} \cdot 0.9) + 5 \cdot 0.9 \cdot (\sqrt{2} \cdot 0.1 + \sqrt{5} \cdot 0.9)) +$$
$$5 \cdot 0.9 \cdot 0.1 \cdot 1 \cdot (\sqrt{2} \cdot 0.1 + \sqrt{5} \cdot 0.9) = 2.993$$

Thus:

$$E[D_B] = \quad 1200(4.7 + 1.853 + 2.993) = 11455.2 \approx 11455$$

- Policy C:
  The BMEMU will be 100. Since there's a full shutdown in the first month:

$$E\left[\sqrt{F_{C_1}}\right] = \quad \sqrt{0.1}$$
$$E\left[F_{C_1}\right] = \quad 0.1$$

  And since months 2 and 3 are NO-OP:

$$E\left[\sqrt{F_{C_2}}\right] = E\left[\sqrt{F_{C_3}}\right] = \quad 2.153$$
$$E\left[F_{C_2}\right] = E\left[F_{C_3}\right] = \quad 4.7$$

  Thus:

$$E[D_C] = \quad 1200\left[E\left[\sqrt{F_{C_1}}\right] + E\left[F_{C_1}\right]\left(E\left[\sqrt{F_{C_2}}\right] + E\left[F_{C_2}\right]E\left[\sqrt{F_{C_3}}\right]\right)\right] =$$
$$1200[\sqrt{0.1} + 0.1(2.153 + 4.7 \cdot 2.153)] = 1852$$

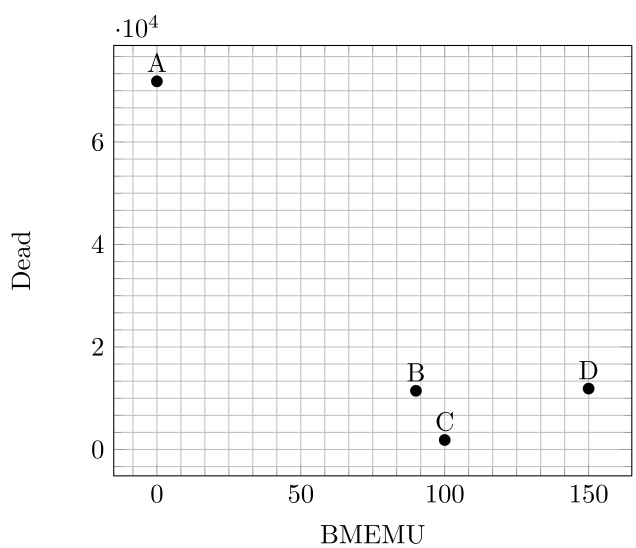
- Policy D:
  The BMEMU will be 150. Since all three months are partial lockdown:

$$E\left[\sqrt{F_{D_j}}\right] = \quad \sqrt{2}$$
$$E\left[F_{D_j}\right] = \quad 2$$

  for all $1 \leq j \leq 3$. And thus:

$$E[D_D] = \quad 1200\left[E\left[\sqrt{F_{D_1}}\right] + E\left[F_{D_1}\right]\left(E\left[\sqrt{F_{D_2}}\right] + E\left[F_{D_2}\right]E\left[\sqrt{F_{D_3}}\right]\right)\right] =$$
$$1200[\sqrt{2} + 2(\sqrt{2} + 2 \cdot \sqrt{2})] = 11879.393 \approx 11879$$

So let's look at the values:



SO we have that D is dominated by both B and C.

b) Utility

Let us calculate the utility values in the described case:

$$\begin{aligned} U(A) &= -(71828 \cdot 0.05 + 0) = -3591.4 \\ U(B) &= -(11455 \cdot 0.05 + 90) = -662.75 \\ U(C) &= -(1852 \cdot 0.05 + 100) = -192.6 \\ U(D) &= -(11879 \cdot 0.05 + 150) = -743.95 \end{aligned}$$

We see that best policy is C due to it having the highest utility.

c) Value of Information

Let us recall the expression for the value of information - Expected value of best action given information minus expected value of best action without information. Since only policy B is affected by the change in infection rate, let us consider it only. So:

$$EVI = E[U(B)|F_{B_1}] - U(B) = 0.9 \cdot U(B|F_{B_1} = 5) + 0.1 \cdot U(B\ |F_{B_1} = 2) - U(B)$$

- In the case of a factor of 5:
  BMEMU will be equal to 100 and:

$$E[D_B\ |F_{B_1} = 5] = 1200\left[\sqrt{5} + 5 \cdot E\left[\sqrt{F_{B_2}}\ \middle|F_{B_1} = 5\right] + 5 \cdot E\left[F_{B_2} \cdot \sqrt{F_{B_3}}\ \middle|F_{B_1} = 5\right]\right]$$

  when:

$$\begin{aligned} E\left[\sqrt{F_{B_2}}\ \middle|F_{B_1} = 5\right] &= \sqrt{0.1} \\ E\left[F_{B_2} \cdot \sqrt{F_{B_3}}\ \middle|F_{B_1} = 5\right] &= 0.1 \cdot E\left[\sqrt{F_{B_3}}\ \middle|F_{B_1} = 5, F_{B_2} = 0.1\right] = \\ &\quad 0.1 \sum_{f_{B_3} \in \{2,5\}} \sqrt{f_{B_3}} \cdot P(F_{B_3} = f_{B_3} \mid F_{B_1} = 5, F_{B_2} = 0.1) \\ &\quad 0.1(\sqrt{2} \cdot 0.1 + \sqrt{5} \cdot 0.9) = 0.215 \end{aligned}$$

  Thus:

$$E[D_B\ |F_{B_1} = 5] = 1200[\sqrt{5} + 5 \cdot \sqrt{0.1} + 5 \cdot 0.215] = 5872.97$$

  So the utility in this case will be:

$$U(B\ |F_{B_1} = 5) = -(5872.97 \cdot 0.05 + 100) = -393.648$$

- In the case of a factor of 2:
  BMEMU will be equal to 0 and:

$$\begin{aligned} E[D_B\ |F_{B_1} = 2] &= 1200\left[\sqrt{2} + 2 \cdot E\left[\sqrt{F_{B_2}}\ \middle|F_{B_1} = 2\right] + 2 \cdot E\left[F_{B_2} \cdot \sqrt{F_{B_3}}\ \middle|F_{B_1} = 2\right]\right] = \\ &\quad 1200\left[\sqrt{2} + 2 \cdot E\left[\sqrt{F_{B_2}}\ \middle|F_{B_1} = 2\right] + 2 \cdot E\left[F_{B_2}\middle|F_{B_1} = 2\right] \cdot E\left[\sqrt{F_{B_3}}\ \middle|F_{B_1} = 2\right]\right] \end{aligned}$$

  when:

$$\begin{aligned} E\left[\sqrt{F_{B_2}}\ \middle|F_{B_1} = 2\right] &= E\left[\sqrt{F_{B_3}}\ \middle|F_{B_1} = 2\right] = 2.153 \\ E\left[F_{B_2}\ \middle|F_{B_1} = 2\right] &= 4.7 \end{aligned}$$

Thus:

$$E[D_B \mid F_{B_1} = 2] = \quad 1200[\sqrt{2} + 2 \cdot 2.153 + 2 \cdot 4.7 \cdot 2.153] = 31162.169$$

So the utility in this case will be:

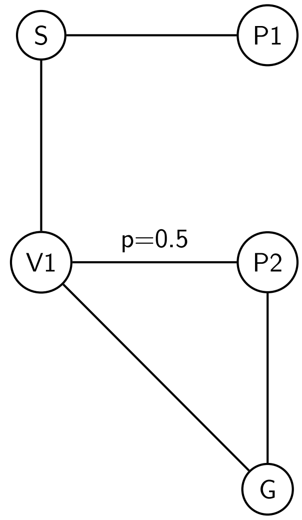
$$U(B \mid F_{B_1} = 2) = -(31162.169 \cdot 0.05 + 0) = -1558.108$$

So we get:

$$EVI = 0.9 \cdot (-393.648) + 0.1 \cdot (-1558.108) - (-662.75) = 152.656$$

5 Markov Decision Problem

We are again presented with the Hurricane Evacuation decision problem on a graph $G = (V, E)$ with the vertices and edges described. Let us draw the graph:



b1) Formalization

To define the problem as a MDP, we must define the belief states, the transition probability function, and the reward function.

- Let us define a belief state as a pair $s = \langle v, b \rangle$ such that $v \in V$ and $b \in \{0, 1, U\}$. v stands for the current location and b stands for the information the state has about the blocked situation of edge $e = (V1, P2)$ when 1 is for when it known that it is blocked, 0 for when it is known that it is not and U is for situations in which we do not know.

- Let us define the transition probabilities $P(s_2|s_1, a)$ - the probability of transitioning from state s_1 to s_2 when the action a is performed:

$$P(\langle \phi, \psi \rangle \mid \langle \zeta, \psi \rangle,\ \zeta \to \phi) = 1\ ;\ \forall \phi, \zeta \in \{S, P1\}, \phi \neq \zeta, \psi \in \{0, 1, U\}$$
$$P(\langle \phi, \psi \rangle \mid \langle \zeta, \psi \rangle,\ \zeta \to \phi) = 1\ ;\ \forall \phi, \zeta \in \{S, V1\}, \phi \neq \zeta, \psi \in \{0, 1\}$$
$$P(\langle V1, \psi \rangle \mid \langle S, U \rangle,\ S \to V1) = 0.5\ ;\ \forall \psi \in \{0, 1\}$$
$$P(\langle \phi, 0 \rangle \mid \langle \zeta, 0 \rangle,\ \zeta \to \phi) = 1\ ;\ \forall \phi, \zeta \in \{V1, P2\}, \phi \neq \zeta$$
$$P(\langle G, \psi \rangle \mid \langle \zeta, \psi \rangle,\ \zeta \to G) = 1\ ;\ \forall \psi \in \{0, 1\}, \zeta \in \{V1, P2\}$$

  and when all other probabilities are 0.

- Reward function:
  Let the reward function be the edge of the transition of the action with a minus sign:

$$R(\langle v, b \rangle, v \to u) = -w(v, u)\ ;\ \forall v, u \in V, b \in \{0, 1, U\}$$

b2) Optimal Policy

Let us find the optimal policy by solving the set of Bellman equations as defined in class:

$$U(S) = \max_a \left\{ R(S, a) + \sum_{S'} U(S')T(S, a, S') \right\}$$

Let us define the utility values of the goal state:

$$U(\langle G, b\rangle) = 0 \;;\; \forall b \in \{0, 1, U\}$$

Let us calculate:

$$\begin{aligned}
U(\langle S, U\rangle) = \max\Big\{ \quad & U\Big(\langle V1, 0\rangle\Big)T\Big(\langle S, U\rangle, S \to V1, \langle V1, 0\rangle\Big) + R\Big(\langle S, U\rangle, S \to V1\Big) + \\
& U\Big(\langle V1, 1\rangle\Big)T\Big(\langle S, U\rangle, S \to V1, \langle V1, 1\rangle\Big) + R\Big(\langle S, U\rangle, S \to V1\Big), \\
& U\Big(\langle P1, U\rangle\Big)T\Big(\langle S, U\rangle, S \to P1, \langle P1, U\rangle\Big) + R\Big(\langle S, U\rangle, S \to P1\Big)\Big\} = \\
\max\Big\{ \quad & 0.5U\Big(\langle V1, 0\rangle\Big) - 1 + 0.5U\Big(\langle V1, 1\rangle\Big) - 1,\ U\Big(\langle P1, U\rangle\Big) - 1\Big\} = \\
\max\Big\{ \quad & 0.5\Big(U\Big(\langle V1, 0\rangle\Big) + U\Big(\langle V1, 1\rangle\Big)\Big) - 2,\ U\Big(\langle P1, U\rangle\Big) - 1\Big\} \\
U(\langle S, 0\rangle) = \max\Big\{ \quad & U\Big(\langle V1, 0\rangle\Big)T\Big(\langle S, 0\rangle, S \to V1, \langle V1, 0\rangle\Big) + R\Big(\langle S, 0\rangle, S \to V1\Big), \\
& U\Big(\langle P1, 0\rangle\Big)T\Big(\langle S, 0\rangle, S \to P1, \langle P1, 0\rangle\Big) + R\Big(\langle S, 0\rangle, S \to P1\Big)\Big\} = \\
\max\Big\{ \quad & U\Big(\langle V1, 0\rangle\Big) - 1,\ U\Big(\langle P1, 0\rangle\Big) - 1\Big\} = \\
\max\Big\{ \quad & U\Big(\langle V1, 0\rangle\Big),\ U\Big(\langle P1, 0\rangle\Big)\Big\} - 1 \leq -1 \\
U(\langle S, 1\rangle) = \max\Big\{ \quad & U\Big(\langle V1, 1\rangle\Big)T\Big(\langle S, 1\rangle, S \to V1, \langle V1, 1\rangle\Big) + R\Big(\langle S, 1\rangle, S \to V1\Big), \\
& U\Big(\langle P1, 1\rangle\Big)T\Big(\langle S, 1\rangle, S \to P1, \langle P1, 1\rangle\Big) + R\Big(\langle S, 1\rangle, S \to P1\Big)\Big\} = \\
\max\Big\{ \quad & U\Big(\langle V1, 1\rangle\Big) - 1,\ U\Big(\langle P1, 1\rangle\Big) - 1\Big\} = \max\Big\{U\Big(\langle V1, 1\rangle\Big),\ U\Big(\langle P1, 1\rangle\Big)\Big\} - 1 \leq -1 \\
U(\langle V1, 0\rangle) = \max\Big\{ \quad & U\Big(\langle G, 0\rangle\Big)T\Big(\langle V1, 0\rangle, V1 \to G, \langle G, 0\rangle\Big) + R\Big(\langle V1, 0\rangle, V1 \to G\Big), \\
& U\Big(\langle P2, 0\rangle\Big)T\Big(\langle V1, 0\rangle, V1 \to P2, \langle P2, 0\rangle\Big) + R\Big(\langle V1, 0\rangle, V1 \to P2\Big), \\
& U\Big(\langle S, 0\rangle\Big)T\Big(\langle V1, 0\rangle, V1 \to S, \langle S, 0\rangle\Big) + R\Big(\langle V1, 0\rangle, V1 \to S\Big)\Big\} = \\
\max\Big\{ \quad & -5,\ U\Big(\langle P2, 0\rangle\Big) - 1,\ U\Big(\langle S, 0\rangle\Big) - 1\Big\} \\
U(\langle V1, 1\rangle) = \max\Big\{ \quad & U\Big(\langle G, 1\rangle\Big)T\Big(\langle V1, 1\rangle, V1 \to G, \langle G, 1\rangle\Big) + R\Big(\langle V1, 1\rangle, V1 \to G\Big), \\
& U\Big(\langle S, 1\rangle\Big)T\Big(\langle V1, 1\rangle, V1 \to S, \langle S, 1\rangle\Big) + R\Big(\langle V1, 1\rangle, V1 \to S\Big)\Big\} = \\
\max\Big\{ \quad & -5,\ U\Big(\langle S, 1\rangle\Big) - 1\Big\} \\
U(\langle P2, 0\rangle) = \max\Big\{ \quad & U\Big(\langle G, 0\rangle\Big)T\Big(\langle P2, 0\rangle, P2 \to G, \langle G, 0\rangle\Big) + R\Big(\langle P2, 0\rangle, P2 \to G\Big), \\
& U\Big(\langle V1, 0\rangle\Big)T\Big(\langle P2, 0\rangle, P2 \to V1, \langle V1, 0\rangle\Big) + R\Big(\langle P2, 0\rangle, P2 \to V1\Big)\Big\} = \\
\max\Big\{ \quad & -1,\ U\Big(\langle V1, 0\rangle\Big) - 1\Big\}
\end{aligned}$$

Since each utility if composed of the values of the reward function which is always negative, we know that the utility values must be negative. Thus we get:

$$U(\langle P2, 0\rangle) = -1$$

Let's go on:

$$\begin{aligned}
U(\langle P2,1\rangle) = \max\Big\{ \quad & U\Big(\langle G,1\rangle\Big)T\Big(\langle P2,1\rangle, P2\rightarrow G, \langle G,1\rangle\Big) + R\Big(\langle P2,1\rangle, P2\rightarrow G\Big), \\
& U\Big(\langle V1,1\rangle\Big)T\Big(\langle P2,1\rangle, P2\rightarrow V1, \langle V1,1\rangle\Big) + R\Big(\langle P2,1\rangle, P2\rightarrow V1\Big)\Big\} = \\
\max\Big\{ \quad & -1,\ U\Big(\langle V1,1\rangle\Big) - 1\Big\} = -1 \\
U(\langle P1,U\rangle) = \max\Big\{ \quad & U\Big(\langle S,U\rangle\Big)T\Big(\langle P1,U\rangle, P1\rightarrow S, \langle S,U\rangle\Big) + R\Big(\langle P1,U\rangle, P1\rightarrow S\Big)\Big\} = \\
& U\Big(\langle S,U\rangle\Big) - 1
\end{aligned}$$

Plugging in the connections we got, let us now observe the set of equations we have:

$$\begin{aligned}
U(\langle S,U\rangle) = &\quad 0.5\Big(-2-5\Big) - 2 = -3.5 \\
U(\langle S,0\rangle) \leq &\quad -1 \\
U(\langle S,1\rangle) \leq &\quad -1 \\
U(\langle V1,0\rangle) = &\quad -2 \\
U(\langle V1,1\rangle) = &\quad -5 \\
U(\langle P1,U\rangle) = &\quad -4-1 = -5
\end{aligned}$$

Thus the optimal policy is:

1. Do $S \rightarrow V1$
2. If e is blocked: Do $V1 \rightarrow G$
3. Else: Do Do $V1 \rightarrow P2$ and then $P2 \rightarrow G$

6 Decision Trees

a) Tree Construction

Let's look at each attribute and its correlation to the decision:

- Attribute A: branch 2 has 2 N and 2 Y, branch 3 also has 2 N and 2 Y
- Attribute B: branch F has 3 N and 1 Y, branch T has 1 N and 3 Y
- Attribute C: branch L has 3 N and 1 Y, branch H has 1 N and 3 Y
- Attribute D: branch 1 has 3 N and 3 Y, branch 0 has 1 N and 1 Y

Let us now calculate the entropy of each attribute:

$$\begin{aligned} H(A) = H(D) = &\quad H\Big(\langle\frac{1}{2},\frac{1}{2}\rangle\Big) + H\Big(\langle\frac{1}{2},\frac{1}{2}\rangle\Big) = 2\cdot 2\cdot\Big(-\frac{1}{2}\log\Big(\frac{1}{2}\Big)\Big) = 2 \\ H(B) = H(C) = &\quad H\Big(\langle\frac{3}{4},\frac{1}{4}\rangle\Big) + H\Big(\langle\frac{3}{4},\frac{1}{4}\rangle\Big) = 2\cdot\Big(-\frac{3}{4}\log\Big(\frac{3}{4}\Big) - \frac{1}{4}\log\Big(\frac{1}{4}\Big)\Big) = 1.622 \end{aligned}$$

So we have that B, C have smaller entropy values than A, D. So let us choose C w.l.o.g. Now let's split into cases with regards to C and consider the rest:

- C = L

  - Attribute A: branch 2 has 1 N and 1 Y, branch 3 has 2 N and 0 Y
  - Attribute B: branch F has 2 N and 0 Y, branch T has 1 N and 1 Y
  - Attribute D: branch 0 has 0 N and 1 Y, branch 1 has 3 N and 0 Y

  So the corresponding attribute values are:

  $$\begin{aligned} H(A) = H(B) = &\quad H\Big(\langle 1,0\rangle\Big) + H\Big(\langle\frac{1}{2},\frac{1}{2}\rangle\Big) = 0 + 2\cdot\Big(-\frac{1}{2}\log\Big(\frac{1}{2}\Big)\Big) = 1 \\ H(D) = &\quad 2\cdot H\Big(\langle 1,0\rangle\Big) = 0 \end{aligned}$$

  Thus we choose D

- C = H

  - Attribute A: branch 2 has 1 N and 1 Y, branch 3 has 0 N and 2 Y
  - Attribute B: branch F has 1 N and 1 Y, branch T has 0 N and 2 Y
  - Attribute D: branch 0 has 1 N and 0 Y, branch 1 has 0 N and 3 Y

  So the corresponding attribute values are:

  $$\begin{aligned} H(A) = H(B) = &\quad H\Big(\langle 1,0\rangle\Big) + H\Big(\langle\frac{1}{2},\frac{1}{2}\rangle\Big) = 1 \\ H(D) = &\quad 2\cdot H\Big(\langle 1,0\rangle\Big) = 0 \end{aligned}$$

  Thus we choose D. The final tree is:
  C: C = L ; D: D = 0 ; decision = Y
  D = 1 ; decision = N
  C = H ; D: D = 0 ; decision = N
  D = 1 ; decision = Y

b) Compact Tree

We cannot build a more compact tree in this case, since choosing B instead of C as the first attribute to classify by results in an even larger tree - which is the only remaining option to choose from given the entropy values we got.

7 Neural Network

a) Hidden Units

We saw in class that a network without a hidden unit cannot implement the XOR function. Fixing all inputs except 2 in the given problem results in the XOR function. Thus this problem cannot be solved by a network without a hidden unit.

b) Building A Network

Let us consider the input layer which is the 7 inputs. After that, we will construct an additional layer with 3 units. These 3 units will have a threshold of ≥ 2, ≥ 4 and ≥ 6. The output layer will hold one unit with a threshold of ≥ 0.5. Let all the input units connect to all the units in both layers with all weights being 1 and let the units in the middle layer connect to the output unit with a weight of -2. Let us go over the cases to see that this works:

- 1 input units: Each of the middle units will produce 0 and thus the output unit will get an input of $1 \geq 0.5$ and thus produce 1.
- 2 input units: The unit ≥ 2 will now get an input of 2 and produce -2, that along with 2 from the inputs - will result in an input of 0 to the output unit and this it will produce 0.
- 3 input units: The unit ≥ 2 will now get an input of 3 and produce -2, that along with 3 from the inputs - will result in an input of 1 to the output unit and this it will produce 1.
- 4 input units 1: The units ≥ 2 and ≥ 4 will now get an input of 4 and produce -4, that along with 4 from the inputs - will result in an input of 0 to the output unit and this it will produce 0.
- 5 input units: The units ≥ 2 and ≥ 4 will now get an input of 5 and produce -4, that along with 5 from the inputs - will result in an input of 1 to the output unit and this it will produce 1.
- 6 input units 1: The units ≥ 2, ≥ 4 and ≥ 6 will now get an input of 6 and produce -6, that along with 6 from the inputs - will result in an input of 0 to the output unit and this it will produce 0.
- 7 input units: The units ≥ 2, ≥ 4 and ≥ 6 will now get an input of 7 and produce -6, that along with 7 from the inputs - will result in an input of 1 to the output unit and this it will produce 1.